The Area Vortex for Modeling Flow through Smoothly Heterogeneous Aquifers

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Motivation

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- Spatial heterogeneity is one of the major remaining hurdles in the development of AEM
- Tiling the domain with inhomogeneities is computationally expensive
- Methods can be developed to model non-piecewise constant heterogeneity without tiling

Outline

- A conversation about modeling continuously variable conductivity

- The multi-quadric area vortex
  - Incremental advance
  - Not particularly useful
- Potential-based formulations
  - Some crazy ideas
  - “stir the pot”

The Multi-quadric Area Vortex: Governing Equations

Water balance (in terms of stream function):

$$\frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial \Psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{k}\frac{\partial \Psi}{\partial y}\right)=0$$

Re-expressed as the Laplacian of the stream function:

$$\frac{\partial^2 \Psi}{\partial x^2}+\frac{\partial^2 \Psi}{\partial y^2}=-N_\Psi=\frac{1}{k}\frac{\partial k}{\partial x}\frac{\partial \Psi}{\partial x}+\frac{1}{k}\frac{\partial k}{\partial y}\frac{\partial \Psi}{\partial y}$$

$$\nabla^2\Psi=\left[\frac{\partial \ln k}{\partial y}Q_x-\frac{\partial \ln k}{\partial x}Q_y\right]$$

Specified

Dependent upon all elements

The Multi-quadric Area Vortex

- Similar in concept to the multi-quadric area sink (Strack & Jankovic 1999)
  - Operates on the stream function rather than discharge potential
  - Laplacian of the stream function (related to curl), rather than Laplacian of the potential function (leakage), is interpolated using multi-quadric radial basis functions

$$N_{\Psi} = \sum a_n r_n$$

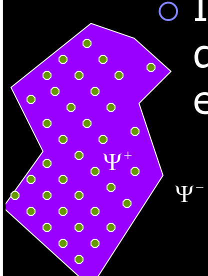
  - Stream function Laplacian, N_{Ψ}, calculated from flow solution and gradients in conductivity at control points:

$$N_{\Psi} = \frac{1}{k}\frac{\partial k}{\partial y} Q_x - \frac{1}{k}\frac{\partial k}{\partial x} Q_y$$

  - Requires modified BCs along border

The Multi-quadric Area Vortex: Internal functional form

- Multi-quadric coefficients (a_n) chosen to meet curl in a least squares sense at a set of control points
- Internal stream function calculated directly from MQ coefficients, external function=0

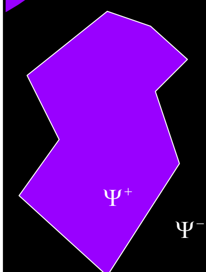


$$\nabla^2 \Psi^{+} = -N_{\Psi}$$

$$\nabla^2 \Psi^{-} = 0$$

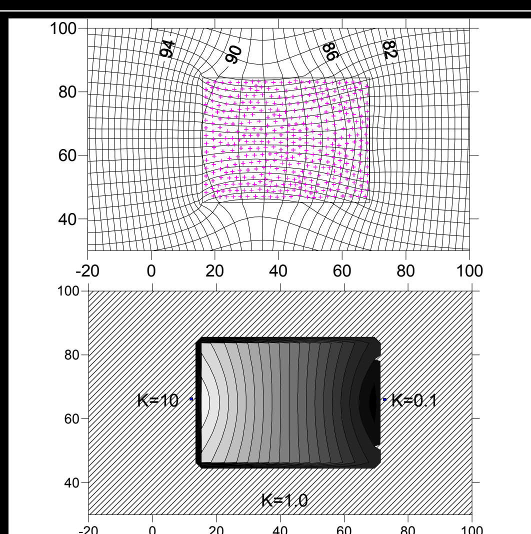
The Multi-quadric Area Vortex: Boundary Conditions

- Uses 3 line element types along border
  - Line doublet:
    - Removes jump in stream function $\Psi^+ = \Psi^-$
  - Line vortex:
    - Removes jump in tangential component of flow due to interior stream function
    - Subject to net curl of zero around vortex border
  - Line dipole:
    - Maintains law of refraction along element border (specific to area vortex for variable k)



$$\frac{Q_t^+}{k^+} = \frac{Q_t^-}{k^-} \quad Q_n^+ = Q_n^-$$

The Multi-quadric Area Vortex: Simple test case



The Multi-quadric Area Vortex: Difficulties

- Since potential is undefined within area vortex, numerical integration is required to calculate head
  - Internal Dirichlet boundaries problematic
- *Many* RBFs required to accurately resolve stream function divergence
  - Singularities in flow or conductivity within AV worsen issues
  - Prohibitive requirement for complex systems
- Can be highly non-linear
  - Iterative solution required
  - Computationally expensive for mildly complex flow systems

Alternative Formulation

- Use an artificial potential w.r.t. reference k:

$$\Phi^* = \begin{cases} \tfrac{1}{2}\bar{k}\phi^2 \\ \bar{k}H^2 - \tfrac{1}{2}\bar{k}H\phi \end{cases}$$

Potential defined!- head available!

- Discharge no longer the derivative of this_ “potential” function ($Q_x = -k' d\Phi^*/dx$; $k = k'\bar{k}$)
- Governing Eqn. in terms of revised potential:

$$\nabla^2\Phi^* = -\frac{\partial Y'}{\partial x}\frac{\partial \Phi^*}{\partial x} - \frac{\partial Y'}{\partial y}\frac{\partial \Phi^*}{\partial y}$$

- Y’ is the perturbation in log conductivity

$$Y' = \ln\left(\frac{k}{\bar{k}}\right) = \ln(k')$$

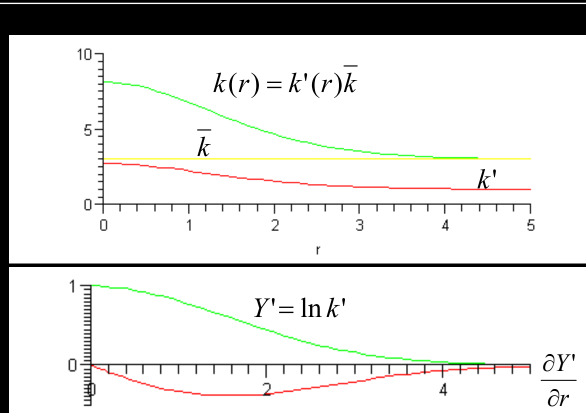
Alternative Formulation: Radial Coordinate System

- Governing Equation:

$$\nabla^2 \Phi^* = -\frac{\partial Y'}{\partial r}\frac{\partial \Phi^*}{\partial r}$$

- Y'(x,y) (and therefore k(x,y)) represented by superposition of radial basis functions
  - Y'=a*F(r) where F(0)=1; F(∞)=0
    - Y'=a*exp(-br^n) (exponential/Gaussian)
    - Y'=a*b/(b+r^n) (Monod/Langmuir)

Alternative Formulation : Radial Coordinate System



Alternative Formulation: Elementary Solutions

- Solution may be superimposed upon standard Harmonic potential solution

$$\nabla^2\Phi^* + \nabla^2\Phi = -\frac{\partial Y'}{\partial r}\frac{\partial(\Phi^* + \Phi)}{\partial r} + 0$$

- Each harmonic element supplemented with additional curl term for each radial basis function
  - Standard solutions must be derived for basic analytic elements
    - Difficulty differs for each RBF Functional form
    - Wells and uniform flow are feasible
    - Line elements may not be amenable to closed-form solution

Alternative Formulation: Advantages

- Potential, and therefore head, still defined
  - Consistent with existing 2D AEM solutions when dk/dx=dk/dy=0;
  - No numerical integration necessary
  - Governing eqn. can be solved with existing MQ area sinks
- Exact- no interpolation necessary, though functional approximations may be required
- Easily extended to 3D
- Potentially fast- added degrees of freedom (DOFs) for the most part directly dependent upon existing DOFs

Alternative Formulation: Disadvantages

- Mathematically difficult
  - Initial derivations indicate that obtaining even simple solutions will be challenging
- Particular solutions are singular for useful RBFs, and will have to cancel out
- Requires M*(N+M+1) degrees of freedom for N harmonic element coefficients and M radial basis functions (memory++)

k(x,y): Other approaches

- Georgitza, 1969; Halek & Svec, 1979
- Define a function $\Phi^* = \sqrt{k'} \cdot \bar{k} H \phi \qquad k(x,y) = k'(x,y)\bar{k}$
- Discharge obtained from $Q_x = -\sqrt{k'}\dfrac{d\Phi^*}{dx} - \Phi^* \dfrac{d\sqrt{k'}}{dx}$

General Inhomogeneous Medium	Helmholtz Inhomogeneous Medium	Harmonic Inhomogeneous Medium	Piecewise Homogeneous Medium
$k'(x,y) = F(x,y)$	$\nabla^2\sqrt{k'} = \alpha^2\sqrt{k'}$	$\nabla^2\sqrt{k'} = 0$	$\sqrt{k'} = 1$
$\nabla^2\Phi^* = \dfrac{\Phi^*}{\sqrt{k'}} \cdot \nabla^2\sqrt{k'}$	$\nabla^2\Phi^* = \alpha^2\Phi^*$	$\nabla^2\Phi^* = 0$	$\nabla^2\Phi^* = \nabla^2\Phi = 0$

Conclusions

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- There is still much to be done to effectively simulate continuously varying properties with the analytic element method
- The multi-quadric area vortex, while conceptually similar to the MQ area sink, suffers from significant drawbacks
- Alternative approaches in terms of artificial potential seem much more promising