A NEW FORMULATION FOR CONJUNCTIVE FLOW IN WETLANDS AND UNDERLYING AQUIFER

M. Gusyev and H. M. Haitjema
School of Public and Environmental Affairs, Indiana; University, Bloomington, IN 47405
Phone: 812-855-0563 Fax: 812-855-7802; mgusyev@indiana.edu - haitjema@indiana.edu

Abstract

An accurate assessment of water and nutrient balances in wetland systems, such as the Florida Everglades, requires conjunctive modeling of flow in these wetlands and underlying aquifers. The South Florida Water Management District (SFWMD) has used the finite difference code MODFLOW with a special Wetlands package in the Everglades to accomplish this. This model treats the wetland flow as laminar with a very high transmissivity that is proportional to the wetland water depth cubed. The MODFLOW solutions appear sensitive to this highly non-linear wetland transmissivity, particularly under conditions of low vegetation density when the wetland conductivity, and thus the transmissivity, is very high. In some cases the model fails to converge on a solution. We propose to formulate the governing differential equation in terms of a pseudo discharge potential instead of potentiometric heads as done in the current MODFLOW model. For the special case of a horizontal wetland bottom the pseudo discharge potential reduces to a true discharge potential. For this case very robust solutions to wetland water elevations and flows are obtained when compared to the formulation in terms of heads. We tested our approach to a few cases of one dimensional flow, both with and without a horizontal wetland bottom. For each case we compared both the robustness and the accuracy of the solution in terms of potentials with that in terms of heads. The results seem promising and may warrant implementation to real world wetland systems.

Introduction

Wetlands have been extensively studied to gain better understanding of their natural functioning. Wetland ecosystems usually are existing for centuries and very sensitive to human influence. The Everglades form the largest wetlands ecosystem in the US and its hydrology patterns and nutrient inputs have been drastically changed by human actions. Recently a major effort is on the way to return many of these wetlands to their natural state. Since hydrology is influencing wetland ecosystems the most, the Everglades restoration plan is focusing on improving South Florida's water management systems. Restrepo et. al. (1998) presented a mathematical model to solve conjunctive wetland and multi-aquifer flow by use of finite-differences and developed a "wetlands package" as a special MODFLOW module. Wilsnack et. al. (2001) used the wetlands package to model water elevations and flows in one of the Everglades wetland areas and reported that modeling results were relatively close to historical data. However, the authors also reported numerical difficulties in wetland areas with low vegetation density. For these cases the resistance to flow in the wetland is very low when compared to that in underlying aquifers. This makes wetland flow very sensitive to small changes in water levels and therefore difficult to solve.

We proposed to reformulate the governing differential equation in terms of discharge potentials and pseudo discharge potentials rather than in terms of potentiometric heads as done in MODFLOW. Tests of a few one dimensional cases of conjunctive wetland and underlying aquifer flow suggest that the (pseudo) discharge potential formulation has significant advantages. The governing differential equation in terms of (pseudo) discharge potentials appears easier to solve (fewer iterations) and leads to stable solutions even when the resistance to flow in the wetlands is very low. In addition, the solution appears more accurate, both in terms of flow and potentiometric heads.

Approach

We will describe flow in the wetland in terms of a discharge vector. The discharge vector in the wetland $\overset{w}{Q}_i$ $[L^2/T]$ is the flux integrated over the wetland depth and per unit length perpendicular to the flow direction (see Figure 1). Kadlec (1990) presented a mathematical framework for describing surface water flow in the wetlands:

$$\overset{w}{Q}_i = -k_w(\phi - h)^\beta \partial_i \phi^\alpha \qquad (i = 1, 2) \tag{1}$$

where ϕ $[L]$ is the water level in the wetland, h $[L]$ the elevation of the wetland bottom or aquifer top, both measured with respect to the base of the aquifer, and k_w $[L^{2-\beta} day^{-1}]$ is a "wetland conductivity". The operator ∂_i denotes differentiation with respect to the coordinate direction x and y. The wetland conductivity is reported by Kadlec (1990) to be in the range of $1.2 * 10^8 \leq k_w \leq 228 * 10^8$. The exponent α $[-]$ depends on the flow characteristics, whereby $\alpha = 1$ for laminar flow, which is assumed here (Restrepo et al., 1998, Wilsnack et al., 2001). The exponent β $[-]$ depends on the wetland vegetation and is usually in the range $2 \leq \beta \leq 4$ (Kadlec, 1990). Following Restrepo et al. (1998) we use $\beta = 3$ and define a wetland transmissivity T_w $[L^2/T]$ as:

$$T_w = k_w(\phi - h)^3 \tag{2}$$

Formulation in terms of heads

In Figure 1 a conceptual cross-section is shown of a wetland connected to an underlying aquifer. The water elevation in the wetland is equal to the head in the aquifer and is denoted by ϕ $[L]$. The wetland bottom or aquifer top h $[L]$ may be constant or vary with location. The net recharge rate due to precipitation is N $[L/T]$. The parameters k_w and k_a are the wetland and aquifer conductivity, respectively. Similarly, $\overset{w}{Q}_i$ and $\overset{a}{Q}_i$ are the discharge vectors in the wetland and aquifer, respectively. The wetlands package

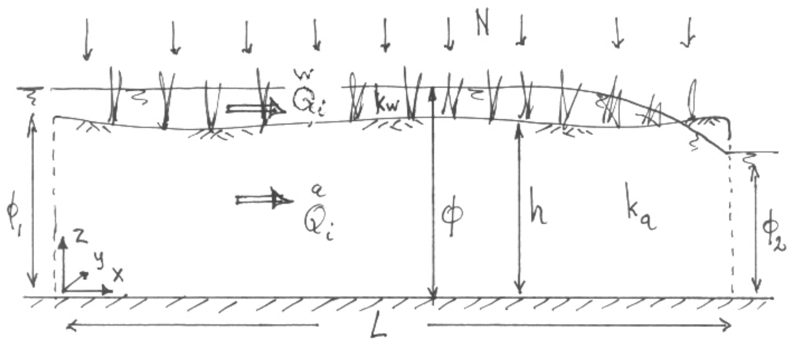

Figure 1: A conceptual cross-section of a wetland connected to an underlying aquifer.

by Restrepo et. al, (1998) treated the flow in the wetlands and underlying soil layer as a single wetlands model layer. It was assumed in the wetlands package that the water level would not drop below the wetland soils in order to avoid periodic drying and rewetting of MODFLOW cells, which leads to computational instabilities.

While we will distinguish between flow in the wetland and aquifer, we will employ only a single grid layer to represent both, thus avoiding the periodic drying and rewetting problems. We also assume that the wetland is hydraulically connected to the underlying aquifer whereby the head in the aquifer equals the water level in the wetland and both are represented by ϕ (see Figure 1). Finally, we restrict ourselves here to isotropic wetland and aquifer properties and steady-state flow. Water balance for the two formulations implies that the divergence of the sum of the discharge vectors in the wetland and the aquifer are equal to the net inflow:

$$\partial_i Q_i = N \qquad (i = 1, 2) \tag{3}$$

where

$$Q_i = \overset{w}{Q}_i + \overset{a}{Q}_i \tag{4}$$

The discharge vector in the wetland $\overset{w}{Q}_i$ is obtained from (1) with $\alpha = 1$ and $\beta = 3$:

$$\overset{w}{Q}_i = -k_w(\phi - h)^3 \partial_i \phi \tag{5}$$

The discharge vector for the aquifer, $\overset{a}{Q}_i$, is defined differently in the zone where the wetland occurs and where it has dried up:

$$\overset{a}{Q}_i = -k_a h \partial_i \phi \qquad \text{wetland present } (\phi > h) \tag{6}$$

$$\overset{a}{Q}_i = -k_a \phi \partial_i \phi \qquad \text{wetland absent } (\phi \le h) \tag{7}$$

Substituting (6) and (5) into (3) yields:

$$\partial_{ii}\phi = -\frac{N + 3k_w(\phi-h)^2\partial_i(\phi-h)\partial_i\phi + k_a\partial_i h\partial_i\phi}{k_w(\phi-h)^3 + k_a h} \quad \text{wetland present } (\phi > h) \tag{8}$$

where $\partial_{ii}\phi$ denotes the Laplacian of ϕ. When the wetland is dry and there is flow in the aquifer only, the differential equation follows from (3) with (7):

$$\partial_{ii}\phi = -\frac{N + k_a\partial_i\phi\partial_i\phi}{k_a\phi} \qquad \text{wetland absent } (\phi \leq h) \tag{9}$$

Formulation in terms of discharge potentials

We propose an alternative formulation. For the case of a horizontal wetland bottom (h = constant), we write the discharge vector Q_i $[L^2/T]$ as the negative gradient of a discharge potential Φ $[L^3/T]$ (Strack, 1989, Haitjema, 1995):

$$Q_i = -\partial_i\Phi \tag{10}$$

where the "comprehensive" discharge potential Φ $[L^3/T]$ is defined as the sum of the discharge potential $\overset{w}{\Phi}$ for the wetland and the discharge potential $\overset{a}{\Phi}$ for the aquifer (see also Strack and Haitjema 1981a):

$$\Phi = \overset{w}{\Phi} + \overset{a}{\Phi} \tag{11}$$

where

$$\overset{w}{\Phi} = \tfrac{1}{4}k_w(\phi-h)^4 \tag{12}$$

and

$$\overset{a}{\Phi} = k_a h\phi - \tfrac{1}{2}k_a h^2 \qquad \text{wetland present } (\phi > h) \tag{13}$$

or

$$\overset{a}{\Phi} = \tfrac{1}{2}k_a\phi^2 \qquad \text{wetland absent } (\phi \leq h) \tag{14}$$

The negative gradients of these potentials are the discharge vectors for wetland flow and aquifer flow respectively:

$$\overset{w}{Q}_i = -\partial_i \overset{w}{\Phi} \tag{15}$$

$$\overset{a}{Q}_i = -\partial_i \overset{a}{\Phi} \tag{16}$$

The differential equation in terms of the comprehensive discharge potential becomes with (3) and (10):

$$\partial_{ii}\Phi = -N \tag{17}$$

Unlike for the formulation in terms of heads, a single differential equation (17) applies to the case where the wetland is wet and the case where it is dry. The comprehensive discharge potential Φ is larger than $\frac{1}{2}k_a h^2$ when the wetland is flooded, that is when $\phi > h$, as is seen from (11) through (14).

The solution procedure using this discharge potential is as follows. First the boundary conditions of the combined wetland and aquifer flow problem are formulated in terms of

Φ. Next, a solution is obtained to (17) that satisfies these boundary conditions. Finally, the wetland water elevation (if indeed flooded) or aquifer head ϕ is obtained from Φ with (11) through (14). For the case of a flooded wetland ($\Phi > \frac{1}{2}k_a h^2$), this leads to a quartic equation in terms of $\phi - h$:

$$(\phi - h)^4 + (4k_a h/k_w)(\phi - h) + 4(\tfrac{1}{2}k_a h^2 - \Phi)/k_a = 0 \tag{18}$$

of which only one root supplies the sought value for ϕ. For the case of flow in the aquifer only ($\Phi \leq \frac{1}{2}k_a h^2$), thus when the wetland is dry, the head follows from (14) as:

$$\phi = \sqrt{\frac{2\Phi}{k_a}} \tag{19}$$

pseudo-discharge potentials

For the cases where the wetland bottom is not horizontal, thus h is a function of location (in general $h(x, y)$), the expressions (10), (15) and (16) are no longer valid. We may continue to use the functions defined by (12) through (14), but the definitions (10), (15) and (16) of the discharge vectors will have to be modified. For flow in the wetland we now have:

$$\overset{w}{Q}_i = -\partial_i \overset{w}{\Phi} - k_w(\phi - h)^3 \partial_i h \tag{20}$$

For flow in the aquifer we have:

$$\overset{a}{Q}_i = -\partial_i \overset{a}{\Phi} + k_a(\phi - h)\partial_i h \qquad \text{wetland present } (\phi > h) \tag{21}$$

or

$$\overset{a}{Q}_i = -\partial_i \overset{a}{\Phi} \qquad \text{wetland absent } (\phi \leq h) \tag{22}$$

The last equation has not changed, since if $\phi < h$ the variation of h is irrelevant. We will refer to the functions $\overset{w}{\Phi}$ and $\overset{a}{\Phi}$ as defined by (12) and (13), respectively, as *pseudo potentials.* In the presence of wetland flow we write:

$$Q_i = \overset{w}{Q}_i + \overset{a}{Q}_i = -\partial_i(\overset{w}{\Phi} + \overset{a}{\Phi}) - k_w(\phi - h)^3 \partial_i h + k_a(\phi - h)\partial_i h \tag{23}$$

Combining (23) with (3) the governing differential equation becomes:

$$\partial_{ii}\Phi = -N - [3k_w(\phi - h)^2 - k_a]\partial_i(\phi - h)\partial_i h - [k_w(\phi - h)^3 - k_a(\phi - h)]\partial_{ii} h \tag{24}$$

We may simplify the right-hand side by limiting us to harmonic functions for $h(x, y)$, so that:

$$\partial_{ii} h = 0 \tag{25}$$

with which (24) reduces to

$$\partial_{ii}\Phi = -N - [3k_w(\phi - h)^2 - k_a]\partial_i(\phi - h)\partial_i h \tag{26}$$

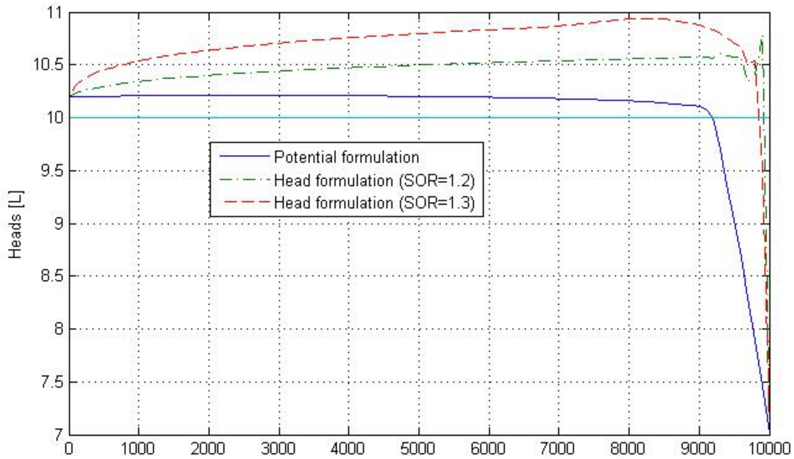

Figure 2: Finite difference solution to one dimensional flow in a wetland and underlying aquifer. The wetland bottom is horizontal.

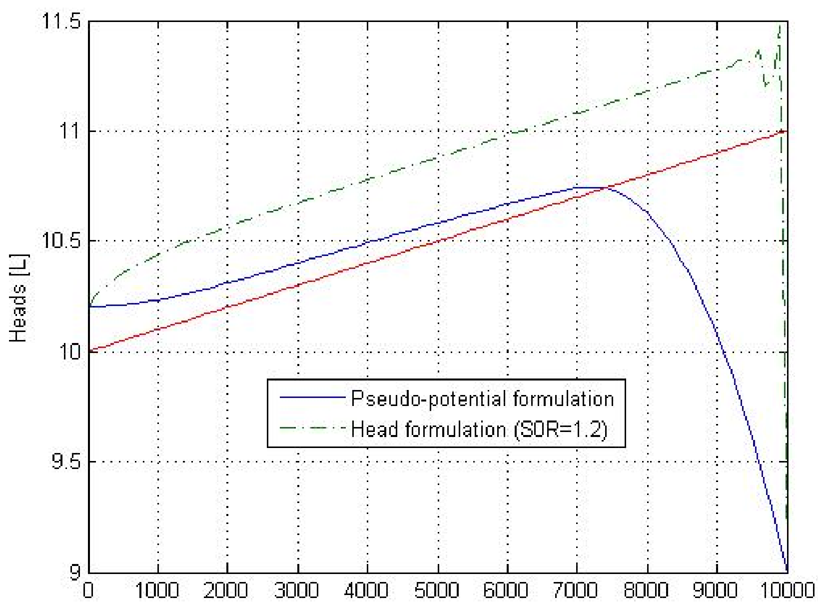

Figure 3: Finite difference solution to one dimensional flow in a wetland and underlying aquifer. The wetland bottom is sloping.

Results of experiments

We compared the two formulations expressed by (8) or (9) and (26) (or (17) when h is constant) using two simple cases of one-dimensional flow. In the case depicted in Figure 2, the wetland bottom is horizontal, while in Figure 3 the wetland bottom is sloping. In both cases the wetland is wet on left-hand side and dry on the right-hand side, see also Figure 1. At the left-hand boundary the water level is kept above the wetland bottom at elevation $\phi_1 = 10.2\ ft$ and at the right-hand boundary the water level is kept below the wetland bottom at elevations $\phi_2 = 7\ ft$ and $\phi_2 = 9\ ft$ for the first and the second problem respectively. The following data have been used: $k_w = 10^8\ 1/ft\,day$, $k_a = 500\ ft/day$, and $N = 0.0023\ ft/day$ (10 $inches/year$). We selected $k_w = 10^8$ because it was for this value that Wilsnak et al. (2001) reported numerical difficulties (failed to solve); they restricted models to cases of $k_w = 10^7\ 1/ft\,day$, which represents more dense vegetation than was actually present. The formulation in terms of heads is based on a cell by cell water balance and is similar to the one in the Wetlands package. To test the relative robustness (stability and accuracy) of the two formulations, we used a basic Gauss-Seidel iterative solution procedure with successive over-relaxation (SOR). The flow problems have been solved by using a finite difference grid of 100 grid lines and applying 100,000 iterations. We generated solutions for various choices of the over-relaxation parameter, but we kept the number of iterations fixed and monitored solution accuracy. We used the maximum over-relaxation parameter value for which a good solution is obtained as a measure of the robustness of the formulation.

We compared the solution in terms of the discharge potentials for the case in Figure 2 (horizontal wetland bottom) to an exact analytic solution, from which it was indistinguishable. There is no known exact solution for the case in Figure 3. However, for the potential formulation the heads satisfy the two boundary conditions on either side of the aquifer exactly, while the difference between the left-hand side and right-hand side of the differential equation (26) is generally less than 10^{-7}, except in the three cells immediately surrounding the transition from wetland flow to flow in the aquifer only where the difference is in the order of 10^{-3} (which is in the order of the recharge). For a smaller recharge rate of $N = 0.00023\ ft/day$ (1 inch per year) we found that the solution is everywhere exact within machine accuracy (order of 10^{-14}). The behavior of the two different formulations in Figure 3 is quite similar as for the case of a horizontal aquifer bottom shown in Figure 2. For both cases we found that the solution in terms of the (pseudo) discharge potential was accurate for all possible values of the relaxation parameter, between 1 and 2. We found that the solution in terms of the head was indistinguishable from the solution in terms of (pseudo) potentials when using a relaxation parameter of 1.1, but it became rapidly inaccurate for larger values. The case for a relaxation parameter of 1.2 is shown in Figure 2 and Figure 3 by the dash-dot line. We also found that the solution in terms of the head failed to converge when we increased the relaxation parameter to 1.6. For both cases, the (pseudo) potential solution appears very robust and converges within 1000 iterations (100 iterations when a relaxation factor of 1.9 is used).

Conclusion

Even for our simple one-dimensional flow problems, we found similar numerical difficulties as reported by Wilsnack et al. (2001) for the far more complex Everglades wetland settings. The differential equation (8) in terms of the water level ϕ in the wetland (equal to the head in the aquifer) is highly non-linear in terms of $(\phi - h)$. This non-linearity is likely cause of the instabilities when seeking numerical solutions for large values of the wetland conductivity k_w. The formulation in terms of heads appeared difficult to solve (only allowing for a very small overrelaxation parameter) and quickly became inaccurate. In contrast, the formulation in terms of discharge potentials performed better in both cases. It was stable for an overrelaxation factor up to 1.9, hence, it required less iterations compared to the formulations in terms of heads.

We suspect, however, that for cases of two-dimensional flow the numerical stability and accuracy of the solution for pseudo potentials is likely dependent on the form of $h(x, y)$. The right-hand side of the differential equation of (26) contains the scalar product of the gradient of the wetland bottom and the gradient of the head $(\partial_i h \partial_i \phi)$, as well as the magnitude of the hydraulic gradient $(\partial_i \phi \partial_i \phi)$. A two dimensional implementation of the pseudo potential formulation may need to explicitly account for the characteristic direction (26) in order to realize a similar solution stability and accuracy as found for the case of one-dimensional flow. We plan to extend our experiments to include two-dimensional flow, resistance to flow between the wetland and underlying aquifer or aquifers, and transient flow.

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