A CONDITION DENOTING FLOW PATTERN CHANGES IN TWO-DIMENISONAL GROUNDWATER FLOW

Wei Jin

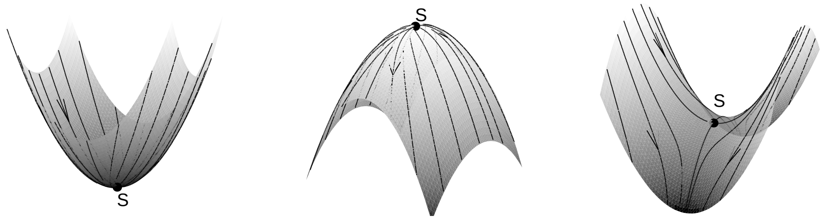
Kansas State University, Department of Agronomy, 2004 Throckmorton Plant Sciences Center, Manhattan, KS 66506-5001, USA; Phone: (785)532-7286; Fax: (785)532-6094; jacking@ksu.edu.

Abstract

The pattern of a groundwater flow is characterized by aquifer features and the number, type, and distribution of stagnation points (locations where the flow rate is zero) in the flow domain. We identify a condition denoting flow pattern changes in two-dimensional groundwater flow obeying Darcy's law by examining changes in stagnation points, using the Taylor series expansion of the discharge vector to represent the flow about such points. We find that the three standard types of stagnation points (minimums, maximums, and saddle points) are completely characterized by the first-order term containing the discharge gradient tensor. However, when the determinant of the tensor becomes zero, stagnation points of other types characterized by higher-order terms may come into existence: they may emerge suddenly, split to a set of new stagnation points, or disappear from the flow, resulting in transitions of flow patterns. Thus, the condition of zero-determinant of the discharge gradient tensor is a condition denoting flow pattern changes. We illustrate the usefulness and significance of this condition in understanding groundwater flows through several examples of steady and transient flows.

1 Introduction

Flow pattern transition is important in classifying and understanding groundwater flow, as evident by previous research by *Tóth* [17,18], *Winter* [19–21], *Anderson and Munter* [1], *Cheng and Anderson* [5], *Nield et al.* [12], *Townley and Trefry* [16], *Smith and Townley* [13], and *Anderson* [2] on surface water - groundwater interaction, and by *Bear and Jacobs* [4], *Javandel and Tsang* [10], *Bakker and Strack* [3], *Steward* [14], *Erdmann* [7], and *Christ and Goltz* [6] on capture zone delineation and well-head protection.



a. Minimum b. Maximum c. Saddle point

Fig. 1. Stagnation points, streamlines, and groundwater head surfaces.

We identify a condition denoting the transition in two-dimensional groundwater flow by examining stagnation points using the Taylor series expansion of the discharge vector. The analysis shows that when the determinant of the gradient of discharge vector becomes zero at a stagnation point, the stagnation point is subject to qualitative changes, leading to a flow pattern transition.

The information provided by this condition may be used to interpret the evolution of groundwater head surfaces for transient flow, to delineate capture zones for steady flow, and to examine flow regime changes for both transient and steady flow. A couple of examples in two-dimensional groundwater flow will be used to illustrate the usefulness.

2 The Condition

The flow in the vicinity of a stagnation point S at time t $[s]$ may be represented by the discharge vector $\mathbf{Q}$ $[m^2/s]$ as

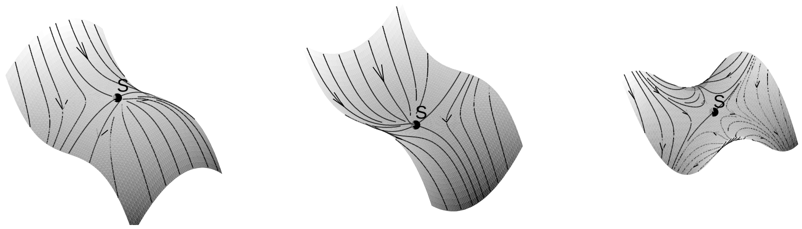
$$\mathbf{Q}(\mathbf{x}) = (\mathbf{x} - \mathbf{x_s}) \cdot (\nabla\mathbf{Q})_\mathbf{s} + \frac{1}{2}(\mathbf{x} - \mathbf{x_s})(\mathbf{x} - \mathbf{x_s}) : (\nabla\nabla\mathbf{Q})_\mathbf{s} + \mathbf{O}(3), \quad (1)$$

where $\mathbf{x_s}$ is the location of S, $\mathbf{O}(3)$ represents terms of order higher than two, and the subscript $\mathbf{s}$ denotes that the term is evaluated at S. The zeroth-order term $\mathbf{Q_s}$ is not listed in the expansion for it is zero according to the definition of a stagnation point, $\mathbf{Q_s} = 0$.

The first-order term is sufficient to approximate the flow about a stagnation point when $\det(\nabla\mathbf{Q})_\mathbf{s} \neq 0$ (e.g., [11], [8]),

$$\mathbf{Q}(\mathbf{x}) = (\mathbf{x} - \mathbf{x_s}) \cdot (\nabla\mathbf{Q})_\mathbf{s}, \quad (2)$$

where $(\nabla\mathbf{Q})_\mathbf{s}$ is the discharge gradient tensor evaluated at point S, and $\det(\nabla\mathbf{Q})_\mathbf{s}$ denotes the determinant of the tensor. The standard minimums, maximums, and saddle points in Figure 1 are examples of such stagnation points in two-dimensional groundwater flow obeying Darcy's law.



a. Saddle-maximum b. Saddle-minimum c. Saddle-saddle

Fig. 2. Examples of higher-order stagnation points.

When $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$, higher-order terms in the Taylor series expansion have to be incorporated in the approximation. This can be demonstrated as follows. The expression (2) can be rewritten in terms of the eigenvalues $\lambda_{\xi s}$ and $\lambda_{\eta s}$ of $(\nabla\mathbf{Q})_\mathbf{s}$,

$$\mathbf{Q}(\mathbf{x}) = \lambda_{\xi s}\mathbf{e}_\xi + \lambda_{\eta s}\mathbf{e}_\eta \tag{3}$$

with $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ as the unit vectors representing the principal directions of $(\nabla\mathbf{Q})_\mathbf{s}$. A relationship exists between the eigenvalues and $\det(\nabla\mathbf{Q})_\mathbf{s}$

$$\lambda_{\xi s}\lambda_{\eta s} = \det(\nabla\mathbf{Q})_\mathbf{s}. \tag{4}$$

The condition $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$ implies that at least one of the two eigenvalues is zero. When one eigenvalue is zero, for example, $\lambda_{\xi s} = 0$, the Taylor series approximation in (1) takes the form

$$\mathbf{Q}(\mathbf{x}) = \lambda_{\eta s}\mathbf{e}_\eta + \left\{\frac{1}{2}[(\mathbf{x}-\mathbf{x}_\mathbf{s})(\mathbf{x}-\mathbf{x}_\mathbf{s}) : (\nabla\nabla\mathbf{Q})_\mathbf{s}]\cdot\mathbf{e}_\xi\right\}\mathbf{e}_\xi. \tag{5}$$

When two eigenvalues are zero, $\lambda_{\xi s} = \lambda_{\eta s} = 0$, the approximation may take the form

$$\mathbf{Q}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}_\mathbf{s})(\mathbf{x}-\mathbf{x}_\mathbf{s}) : (\nabla\nabla\mathbf{Q})_\mathbf{s}. \tag{6}$$

Figure 2 depicts three examples of higher-order stagnation points.

The above analysis demonstrates that the condition $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$ makes a stagnation point special, through which flow pattern transition may occur. Such a stagnation point is referred to as a zero $\det(\nabla\mathbf{Q})_\mathbf{s}$ stagnation point here. Therefore, we may use this condition to quantify and understand flow pattern transition. For a two-dimensional groundwater flow, this condition is equivalent to $\mathbf{Q}_\mathbf{s} = \mathbf{0}$ and $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$;

$$(Q_x)_s = 0, \tag{7a}$$

$$(Q_y)_s = 0, \tag{7b}$$

and

$$\left(\frac{\partial Q_x}{\partial x}\right)_s\left(\frac{\partial Q_y}{\partial y}\right)_s - \left(\frac{\partial Q_x}{\partial y}\right)_s\left(\frac{\partial Q_y}{\partial x}\right)_s = 0. \tag{7c}$$

Several examples of well flow will be used next to illustrate the usefulness and significance of this condition. In these examples the discharge vector is

$$Q_x = Q_0 \cos\theta - \sum_{i=1}^{N} \frac{Q_i}{2\pi} \frac{x - x_{wi}}{(x - x_{wi})^2 + (y - y_{wi})^2}, \tag{8a}$$

$$Q_y = Q_0 \sin\theta - \sum_{i=1}^{N} \frac{Q_i}{2\pi} \frac{y - y_{wi}}{(x - x_{wi})^2 + (y - y_{wi})^2}, \tag{8b}$$

for a system of N steady wells, where Q_0 and θ are the magnitude and direction of the regional flow, and Q_i and $(x_{wi},\, y_{wi})$ are the pumping rate and location of the ith well [15,9]. For a system of transient wells pumping for a duration of t_0,

$$Q_x = Q_0 \cos\theta - \sum_{i=1}^{N} \frac{Q_i}{2\pi} \frac{x - x_{wi}}{(x - x_{wi})^2 + (y - y_{wi})^2} F_i(x, y, t), \tag{9a}$$

$$Q_y = Q_0 \sin\theta - \sum_{i=1}^{N} \frac{Q_i}{2\pi} \frac{y - y_{wi}}{(x - x_{wi})^2 + (y - y_{wi})^2} F_i(x, y, t), \tag{9b}$$

where

$$F_i(x, y, t) = e^{-\frac{(x - x_{wi})^2 + (y - y_{wi})^2}{4\alpha_h t}}\, \mathrm{H}(t) - e^{-\frac{(x - x_{wi})^2 + (y - y_{wi})^2}{4\alpha_h (t - t_0)}}\, \mathrm{H}(t - t_0), \tag{10}$$

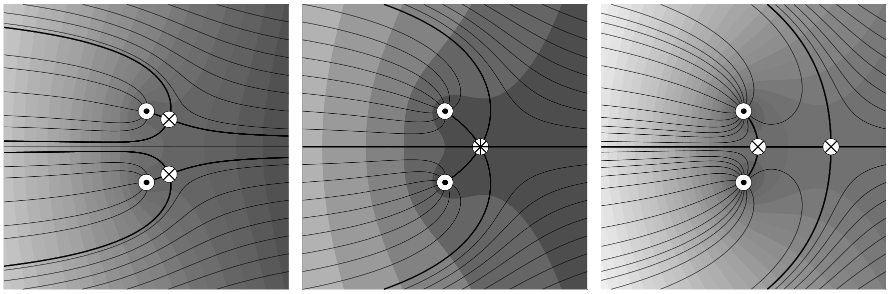
where α_h is the aquifer diffusivity and $\mathrm{H}(t)$ the Heaviside function.

3 Examples

For the problem of two discharge wells depicted in Figure 3, the critical condition $Q/(2\pi Q_0 d) = 1$ where d is the half distance between the two wells, is obtained the condition $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$. While the result is identical to that of the previous studies (e.g., [10]), the new approach is more general and can be used for scenarios where explicit solutions are not available. For example, results are presented in Figure 4 for four, six, eight, and ten wells.

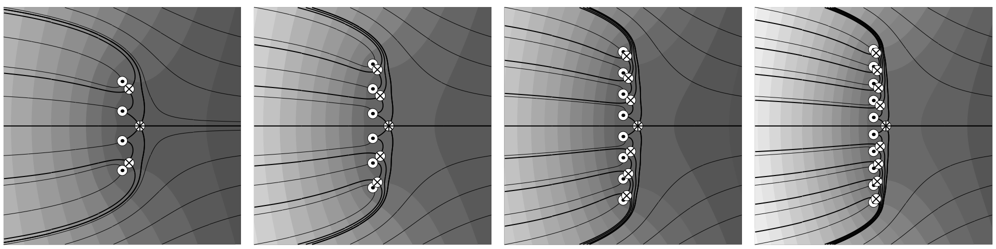
Figure 5 illustrates how the transient flow evolves for the scenario of a single well pumping for a duration of τ_0, where $\tau_0 = t_0/T$ with $T = Q^2/(4\pi^2 Q_0 \alpha_h)$. The flow pattern transition occurs at $\tau_c \approx= 1.25\tau_0$, denoted by the occurrence of a $\det(\nabla\mathbf{Q})_\mathbf{s} = 0$ stagnation point.

Figure 6 depicts trajectories of instantaneous stagnation points for different τ_0's. It shows that smaller the value of τ_0 the longer the normalized time τ/τ_0 it takes for the saddle point and the minimum to coincide to disappear from the domain after pumping stops. For example, $\tau_c/\tau_0 \approx 1.25$ when $\tau_0 = 0.2$, while $\tau_c/\tau_0 \approx 1.02$ when $\tau_0 = 5.0$.



a. $Q/(2\pi Q_0 d) < 1$ b. $Q/(2\pi Q_0 d) = 1$ c. $Q/(2\pi Q_0 d) > 1$

Fig. 3. Steady flow of two discharge wells in regional flow: $\odot$ = discharge wells; $\otimes$ = saddle points; $\circledast$ = zero $\det(\nabla \mathbf{Q})_\mathbf{s}$ stagnation points; solid lines = streamlines; background shadings = head contours.



a. $Q/(2\pi Q_0 d) =$ 0.824 b. $Q/(2\pi Q_0 d) =$ 0.765 c. $Q/(2\pi Q_0 d) =$ 0.735 d. $Q/(2\pi Q_0 d) =$ 0.717

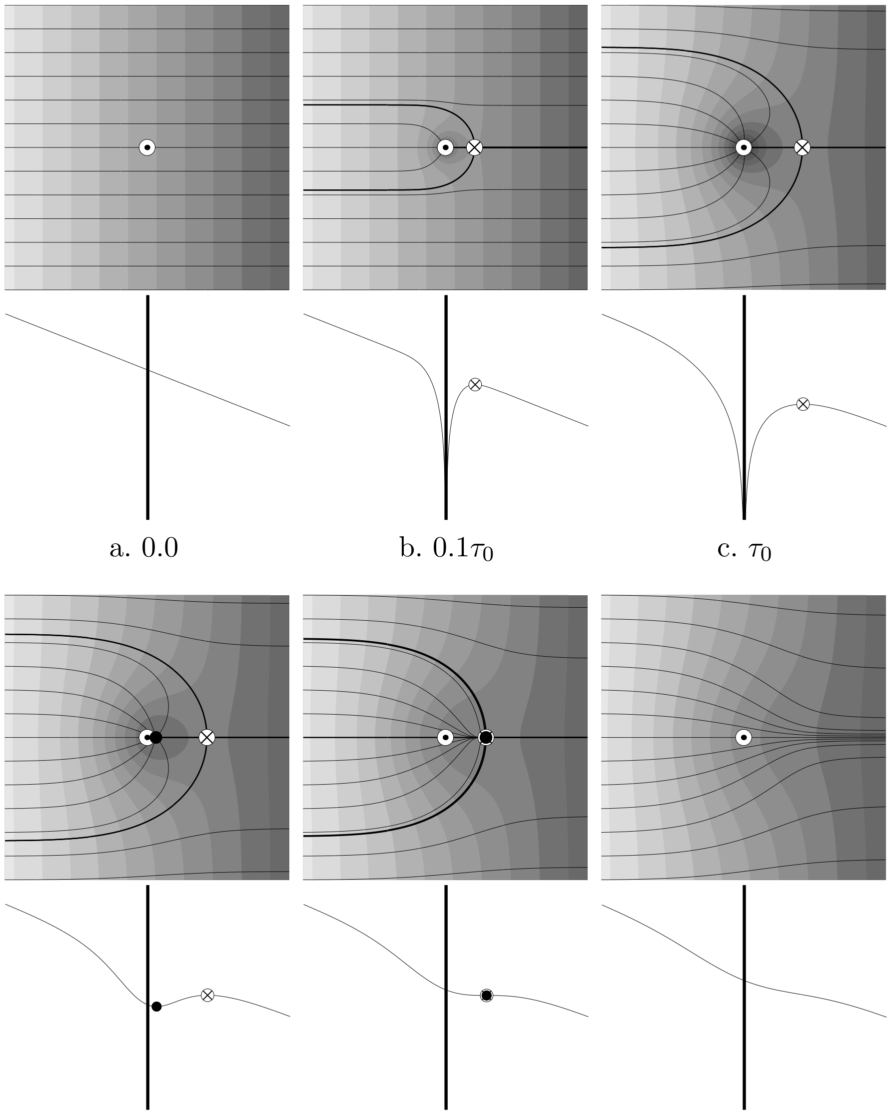
Fig. 4. Steady flow of multiple discharge wells in regional flow: $\odot$ = discharge wells; $\otimes$ = saddle points; $\circledast$ = zero $\det(\nabla \mathbf{Q})_\mathbf{s}$ stagnation points; solid lines = streamlines; background shadings = head contours.

The evolution of two transient discharge wells is depicted in Figure 7, whose steady counterpart is in Figure 3c, $Q/(2\pi Q_0 d) > 1$. Four zero $\det(\nabla \mathbf{Q})_\mathbf{s}$ stagnation points occur during the whole process, depicted in Figure 7c, f, i, and k.

4 Conclusions

This paper identifies a condition for the transition of flow patterns in two-dimensional groundwater flow obeying Darcy's law by examining stagnation points using the Taylor series expansion. The Taylor series expansion of the discharge vector is used to represent flow in the vicinity of a stagnation point. It is found that the condition of $\det(\nabla \mathbf{Q})_\mathbf{s} = 0$ at a stagnation point represents a condition through which flow patterns may change.

Examples of both steady and transient flow have been used to demonstrate the significance and usefulness of this condition in understanding and classifying two-dimensional groundwater flow. Critical parameters were obtained



a. 0.0 b. $0.1\tau_0$ c. τ_0

d. $1.1\tau_0$ e. $\tau_c \approx 1.25\tau_0$ f. $1.5\tau_0$

Fig. 5. Transient flow of a single well in regional flow at various times τ: $\tau_0 = 0.2$; $\odot$ = discharge well; $\bullet$ = minimum; $\otimes$ = saddle point; solid lines = streamlines; background shadings = head contours. A $\det(\nabla \mathbf{Q})_{\mathbf{s}} = 0$ stagnation point is formed in Figure e. Each subfigure contains a plan view and a section view at the well in the regional flow direction.

for pump-and-treat remediation by a system of wells aligned perpendicular to the regional flow by identifying zero $\det(\nabla \mathbf{Q})_{\mathbf{s}}$ stagnation points. Evolution of transient well flow was quantified in terms of critical instants when zero $\det(\nabla \mathbf{Q})_{\mathbf{s}}$ stagnation points occur.

The condition of zero $\det(\nabla \mathbf{Q})_{\mathbf{s}}$ stagnation points is general for flow pattern transition and can be used to investigate transitions for flows with other aquifer features.

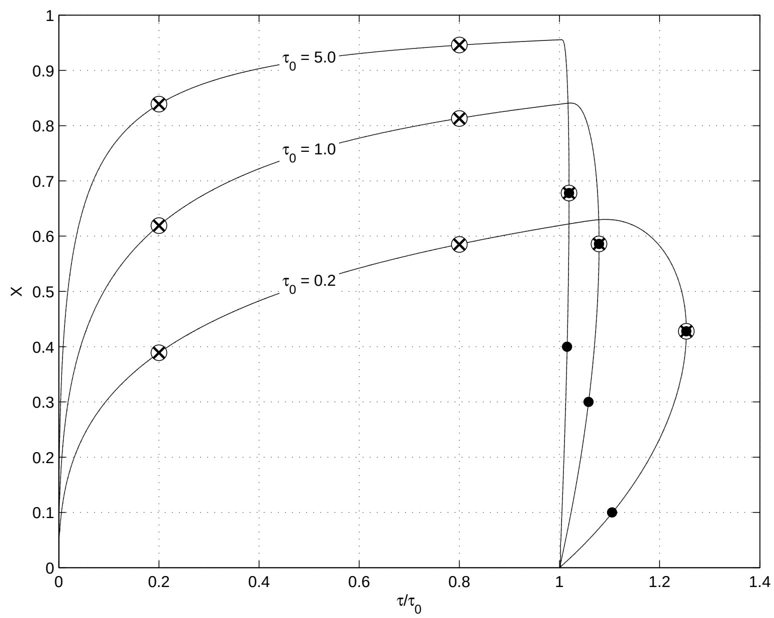


Fig. 6. Trajectories of stagnation points for a variety of τ_0 values: $\otimes$ = saddle points; $\bullet$ = minimums; $\circledcirc$ = zero $\det(\nabla \mathbf{Q})_\mathbf{s}$ stagnation points formed by coincidence of a saddle and a minimum.

Acknowledgments

The author gratefully acknowledge financial support provided by the National Science Foundation (grants NSF19255-KAN19256 and NSF32125-KAN32126) and from the Provost Office's Targeted Excellence Program at Kansas State University.

References

[1] Anderson MP, Munter JA. Seasonal reversals of groundwater flow around lakes and the relevance to stagnation points and lake budget. Water Resources Research 1981. 17(4): 1139-1150.

[2] Anderson, EI. An analytic solution representing groundwater-surface water interaction. Water Resources Research 2002; 39(3): 1071-1078.

[3] Bakker M, Strack ODL. Capture zone delineation in two-dimensional groundwater flow models. Water Resources Research 1996; 32(5): 1309-1315.

[4] Bear J, Jacobs M. On the movement of water bodies injected into aquifers. Journal of Hydrology 1965; 3: 37-57.

[5] Cheng X, Anderson MP. Simulating the influence of lake position on groundwater fluxes. Water Resources Research 1994. 30(7): 2041-2049.

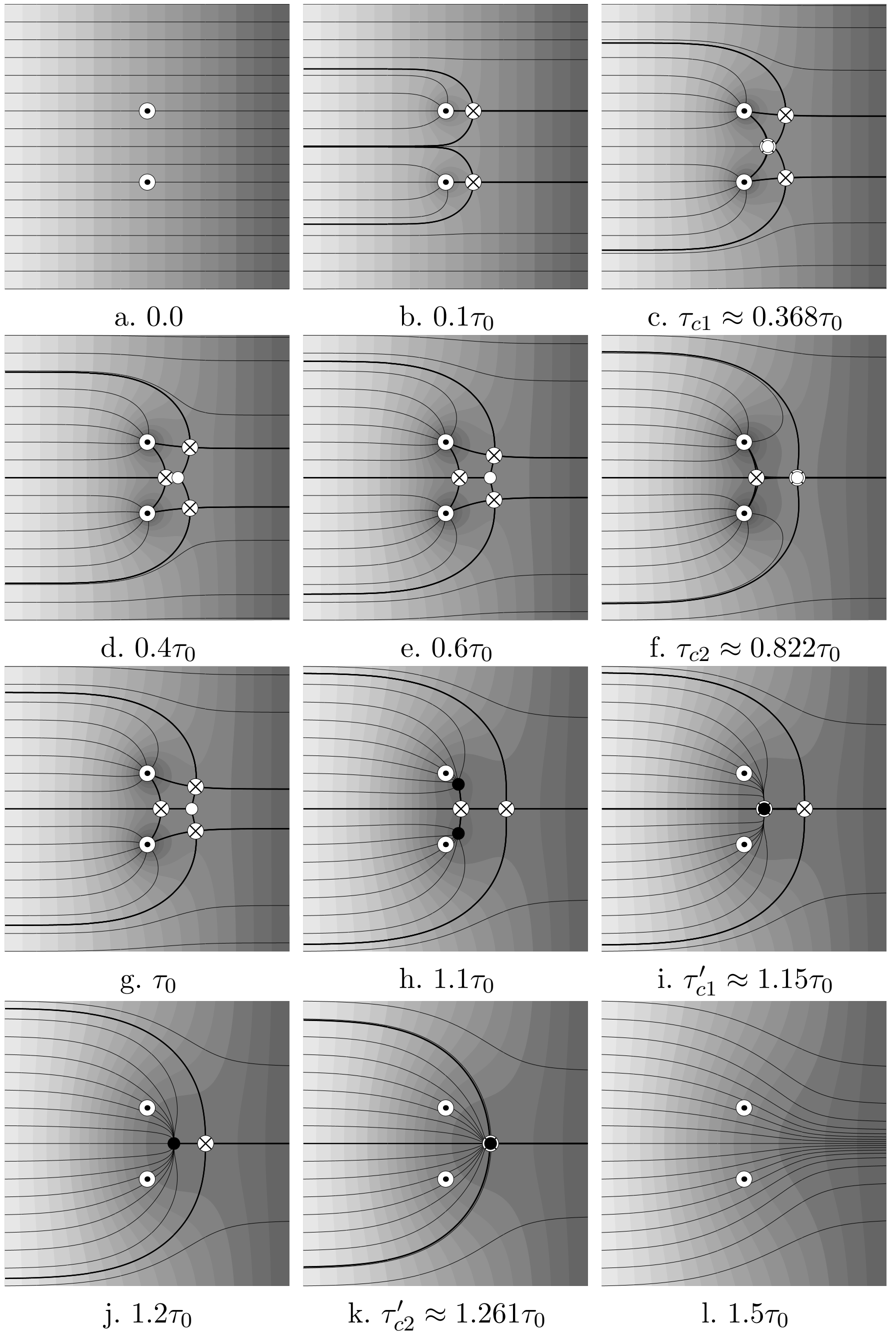


Fig. 7. Transient flow of two discharge wells in regional flow at various times τ: $Q/(2\pi Q_0 d) = 2$; $\tau_0 = (\alpha_h t_0)/L^2 = 0.4$; $\odot$ = discharge wells; $\circ$ = maximums; $\bullet$ = minimums; $\otimes$ = saddle points; solid lines = streamlines; background shadings = head contours. Zero $\det(\nabla \mathbf{Q})_{\mathbf{s}}$ stagnation points appear in c, f, i, and k.

[6] Christ JA, Goltz MN. Hydraulic containment: analytical and semi-analytical models for capture zone curve delineation. Journal of Hydrology 2002; 262: 224-244.

[7] Erdmann J. On capture width and capture zone gaps in multiple-well systems. Ground Water 1999; 38(4): 497-504.

[8] Fienen MN, Luo J, and Kitanidis PK. Semi-analytical homogeneous anisotropic capture zone delineation. Journal of Hydrology 2005; 312: 29-50.

[9] Haitjema HM. Analytic element modeling of groundwater flow. Academic Press, Inc., 1995.

[10] Javandel I, Tsang CF. Capture-zone type curves: A tool for aquifer cleanup. Ground Water 1986; 24(5): 616-625.

[11] Li WH. Differential equations of hydraulic transients, dispersion, and groundwater flow. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1972.

[12] Nield SP, Townley LR, and Barr AD. A framework for quantitative analysis of surface water-groundwater interaction: Flow geometry in a vertical section. Water Resources Research 1994; 30(8): 2461-2475.

[13] Smith AJ, Townley LR. Influence of regional setting on the interaction between shallow lakes and aquifers. Water Resources Research 2002; 38(9): 1170.

[14] Steward DR. Three-dimensional analysis of the capture of contaminated leachate by fully penetrating, partially penetrating, and horizontal wells. Water Resources Research 1999; 35(2): 461-468.

[15] Strack ODL. Groundwater mechanics. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1989.

[16] Townley LR, Trefry MG. Surface water-groundwater interaction near shallow circular lakes: Flow geometry in three dimensions. Water Resources Research 2000; 36(4): 935-949.

[17] Tóth JA. A theory of ground-water motion in small drainage basins in central Alberta, Canada. Journal of Geophysical Research 1962; 67: 4375-4387.

[18] Tóth JA. A theoretical analysis of ground-water flow in small drainage basins. Journal of Geophysical Research 1963; 68: 4795-4811.

[19] Winter TC. Numerical simulation analysis of the interaction of lakes and ground water. U.S. Geological Survey Professional Paper 1976; 1001: 1-40.

[20] Winter TC. Numerical simulation of steady state three-dimensional groundwater flow near lakes. Water Resources Research 1978; 4(2): 245-254.

[21] Winter TC. Recent advances in understanding the interaction of groundwater and surface water. U.S. National Report to IUGG 1991-1994, Rev. Geophys. 1995; 33: 985-994.