The Method of Images for Poisson Problems

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Overview

- Describe a method for finding and using image functions, that is applicable to problems governed by Poisson's equation

$$\nabla^2\Phi = \gamma$$

- Difficulties
  - The image of Φ cannot be added in general, because that would change the specified divergence
  - Conformal mapping of Φ is not generally used because it can deform the distribution of γ
  - Milne-Thomson (1973), gives a special solution for a circular boundary with rotational flow that is analagous to what will be presented here.
- Follow the method of images for a 2-sided boundary described in Strack (1989)

Setting and Boundary Conditions

- Setting:
  - An infinite shallow confined aquifer with piecewise constant properties.
  - There exists a complex discharge potential, $\Omega = \Phi + i\Psi$.
  - There is a closed boundary $\mathcal{B}$, perhaps extending to ∞.
- Boundary conditions along a two sided boundary $\mathcal{B}$

$$\text{Continuity of head:} \qquad \phi^- = \phi^+$$
$$\text{Continuity of flow:} \qquad Q_n^- = Q_n^+$$

- Boundary conditions expressed in terms of $\Phi = kH\phi$

$$\text{Continuity of head:} \qquad \Phi^- = \frac{k^-}{k^+}\Phi^+$$
$$\text{Continuity of flow:} \qquad \partial_n\Phi^- = \partial_n\Phi^+$$

Consider a relatively simple problem . . .

- There is non-constant infiltration created by a real valued function G
$$G = -\sigma(z-d)^2(\overline{z-d})^2$$
- There is a circular boundary, with center $z_c \neq d$, and radius R. The exterior and interior domains, $\mathcal{D}^e$ and $\mathcal{D}^i$, have distinct hydraulic conductivities: k^e and k^i, which require a jump in the real potential Φ
- We seek an analytic element $\overset{E}{\Omega}$ that will satisfy both boundary conditions when added to G.

Approach

- Find an analytic function $\widetilde{G}$ whose real and imaginary parts are exactly equal to G on the boundary.
- Map the boundary to the unit circle in the Z–plane.

$$z = RZ + z_c, \qquad D = Z(d)$$

$$G = -\sigma R^4 (Z - D)^2 (\overline{Z - D})^2$$

- The boundary can be defined by a function $\beta(z) = \overline{z}$. In this case $\overline{Z} = 1/Z$. Substitute $\beta(z)$ for $\overline{Z}$ in G

$$\boxed{\widetilde{G} = -\sigma R^4 \Big(Z - D\Big)^2 \Big(\frac{1}{Z} - \overline{D}\Big)^2}$$

Interesting Properties of $\widetilde{G}$

- $\widetilde{G}(Z)$ is analytic because it is a function of Z only
  - $\widetilde{G}$ does not violate the governing differential equation
  - $\widetilde{G}$ is unique because its real and imaginary parts are fully defined on $\mathcal{B}$
- $\widetilde{G}$ generates no flow across B
  - Why? Because the imaginary part of $\widetilde{G}$ is a constant 0 on $\mathcal{B}$, so the boundary is a streamline.
- $\widetilde{G}$ consists of singularities imaged across $\mathcal{B}$.
  - Hence: "The Method of Images"

Useing $\widetilde{G}$ to create an Analytic Element for the Boundary

- Define a jump function, $\overset{1}{F}$ for the boundary, using $\widetilde{G}$

$$\overset{1}{F}{}^{e} = \widetilde{G} \qquad\qquad z \in \mathcal{D}^e$$
$$\overset{1}{F}{}^{i} = 0 \qquad\qquad z \in \mathcal{D}^e$$

- Boundary conditions
  - The jump is exactly equal G, so $\alpha\overset{1}{F}$ is capable of exactly meeting the continuity of head boundary condition
  - Continuity of flow is met because the normal component of flow is 0 on both sides of $\mathcal{B}$.
- *But*, $\widetilde{G}$ must have singularities in $\mathcal{D}^e$ that have to be removed.

Removing the singularities in $\mathcal{D}^e$ with function $\overset{2}{F}$

In this example, the singularities can be removed algebraically

- Expand $\widetilde{G}$ in a power series to isolate the singularities

$$\widetilde{G} = \sum_{n=-2}^{2} a_n Z^N$$

- In general the singularities should be removed with a function of z to ensure continuity of flow. In this case the mapping is trivial, and we can safely subtract the offending terms as functions of Z.

$$-\overset{2}{F} = -\frac{1}{2}a_0 - \sum_{n=1}^{2} a_n Z^N$$

- It is convenient, but not essential, to subtract one half of the constant term to retain symmetry.

The Analytic Element is the sum of $\overset{1}{F}$ and $\overset{2}{F}$

$$\Omega^e = \alpha\left(\overset{1}{F} - \overset{2}{F}\right)$$

$$= \alpha\left[\sigma R^4\left(D^2 Z^{-2} - 2D(D\overline{D} + 1)Z^{-1} + \frac{1}{2}(1 + 4D\overline{D} + D^2\overline{D}^2)\right)\right]$$

$$\Omega^i = -\alpha\overset{2}{F}$$

$$= -\alpha\left[\sigma R^4\left(\overline{D}^2 Z^2 - 2\overline{D}(D\overline{D} + 1)Z + \frac{1}{2}(1 + 4D\overline{D} + D^2\overline{D}^2)\right)\right]$$

Strength of the element

- The strength of the element is found to vary between -2 and +2

$$\alpha = 2\frac{k^- - k^+}{k^- + k^+}$$

- At the extremes, it creates a constant head boundary, or an impermeable boundary
- At intermediate values, it creates a jump in Φ suitable for an inhomogeneity boundary.

More about $\widetilde{G}$ and $\overset{2}{F}$

- Since $\widetilde{G}$ is analytic, it is possible to derive it after performing a conformal mapping of G – We don't care how the divergence of G maps.
- $\overset{2}{F}$ must be defined as $\overset{2}{F}(z)$ in order to guarantee that it is continuous across $\mathcal{B}$ and throughout both domains.
- There may be multiple ways to extract $\overset{2}{F}$ from $\widetilde{G}$.

More about $\widetilde{G}$

- If G is defined in terms of x and y, which is typical for divergence specified functions, Then $G(z, \overline{z})$ can be derived using these substitutions:

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2}$$

- If G is not strictly real, the image set for $\overset{1}{F}$ can still be derived.
  - Derive $\widetilde{G}$ from $\overline{G}$
  - On $\mathcal{B}$, $\widetilde{G} = \overline{G}$
  - $(\widetilde{G} + G)$ has the required properties on $\mathcal{B}$ that $\widetilde{G}$ alone has when G is strictly real

References

- Milne-Thomson, L. M. 1973. *Theoretical Aerodynamics*. New York, Dover Publications.
- Strack, O.D.L., 1989. *Groundwater Mechanics*. Prentice Hall.