Modeling multi-layer flow and unsaturated flow: AEM solutions to the modified Helmholtz equation

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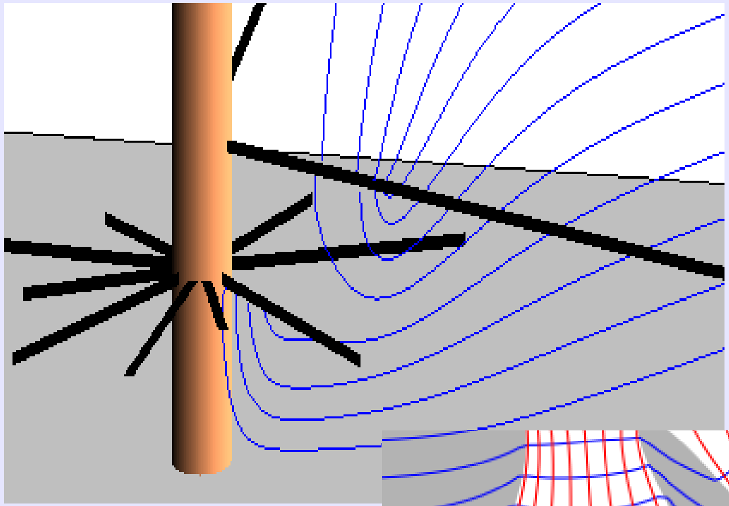
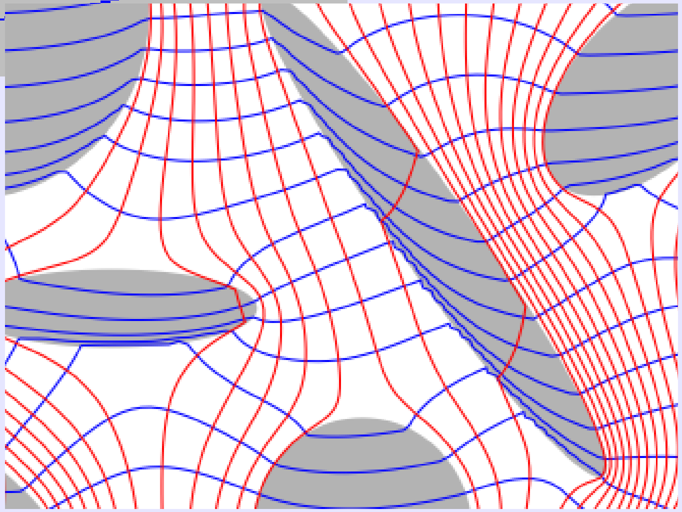




Sabbatical funding:
TU Delft Grants Program

Objective: Model multi-aquifer and multi-layer flow with an analytic, mesh free approach

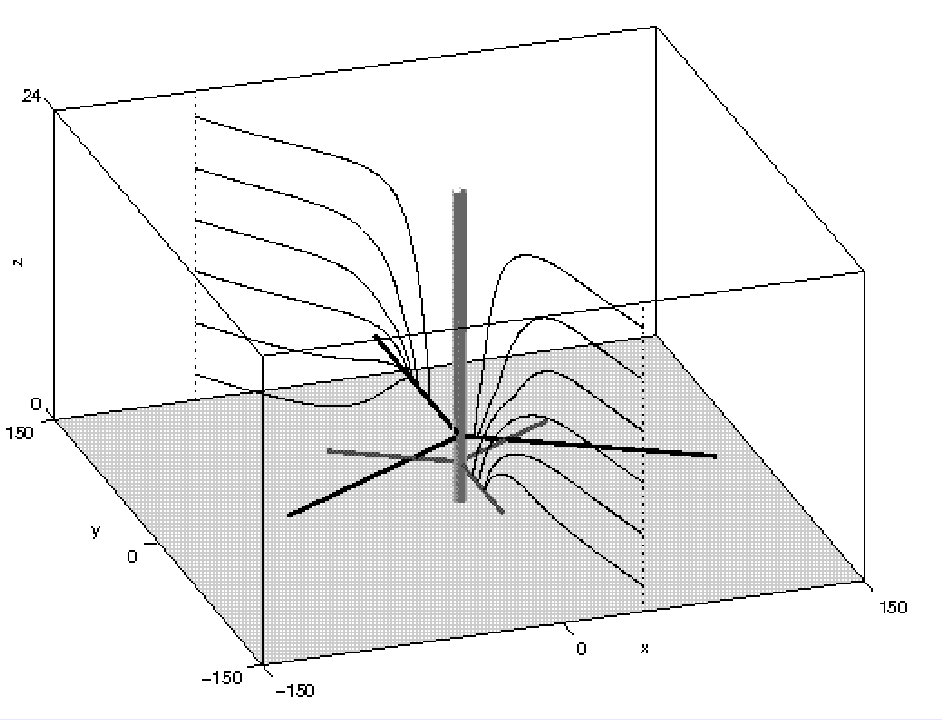
Flow is at steady-state

Resistance to vertical flow is neglected within an aquifer

Head is function of x and y, but flow is 3D

Flow in leaky layers is vertical

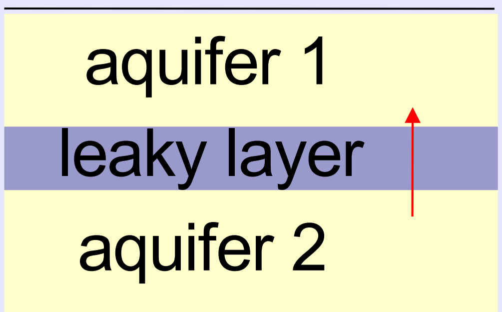
Aquifer properties are piece-wise constant



Funding: US EPA, WHPA, WBL, Brabant Water, Artesia, Amsterdam Water Supply
Collaborators: Otto Strack, Vic Kelson, Ken Luther

Vertical flow between aquifers or layers is computed with Darcy's law

multi-aquifer flow



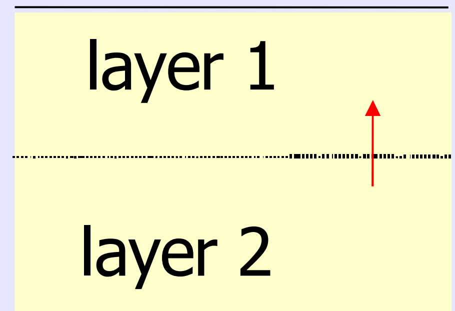
$$q_z = k_z \frac{h_2 - h_1}{d} = \frac{h_2 - h_1}{c}$$

k_z : vertical hydraulic conductivity of leaky layer

d : thickness of leaky layer

$c = d / k_z$: resistance

multi-layer flow



$$q_z = k_z \frac{h_2 - h_1}{(H_1 + H_2)/2} = \frac{h_2 - h_1}{c}$$

k_z : vertical hydraulic conductivity of layers

H_1, H_2: thicknesses of layers

$c = (H_1 + H_2)/(2k_z)$

Flow in 3-layer system is governed by 3 linked differential equations

General form of deq: $\nabla^2 h = E/T$

$$\nabla^2 h_1 = \frac{h_1 - h_2}{c_1 T_1}$$

$$\nabla^2 h_2 = \frac{h_2 - h_1}{c_1 T_2} + \frac{h_2 - h_3}{c_2 T_2}$$

$$\nabla^2 h_3 = \frac{h_3 - h_2}{c_2 T_3}$$

h: hydraulic head (m)

E: sink term (m/d)

$T=kH$: transmissivity (m^2/day)

c: resistance between layers (days)

System of differential equations may be written in matrix form of modified Helmholtz equation

multi-layer theory of Hemker (1984)

$$\nabla^2 \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{c_1 T_1} & \dfrac{-1}{c_1 T_1} & 0 \\ \dfrac{-1}{c_1 T_2} & \dfrac{1}{c_1 T_2} + \dfrac{1}{c_2 T_2} & \dfrac{1}{c_2 T_2} \\ 0 & \dfrac{-1}{c_2 T_3} & \dfrac{1}{c_2 T_3} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

Hermann von Helmholtz

$$\nabla^2 \vec{h} = \mathbf{A} \vec{h}$$

$\mathbf{A}$ is the system matrix and is function of aquifer properties

General form of solution for confined system with N layers: Laplace part and Helmholtz part

General form of solution: $$\vec{h} = h_L \vec{e} + \sum_{n=1}^{N-1} h_n \vec{u}_n$$

h_L fulfills Laplace's deq: $$\nabla^2 h_L = 0$$

h_n fulfills modified Helmholtz deq: $$\nabla^2 h_n = w_n h_n$$

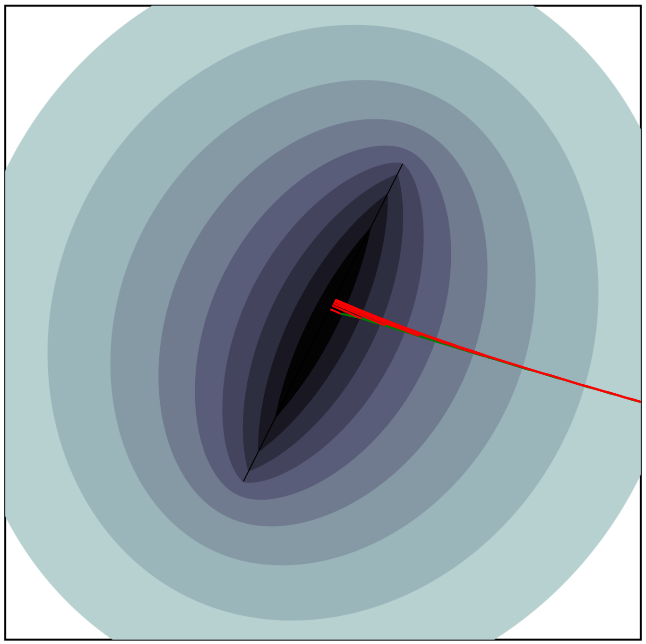
$\vec{e}$ unit vector
$\vec{u}_n$ eigenvector n of system matrix
w_n eigenvalue n of system matrix

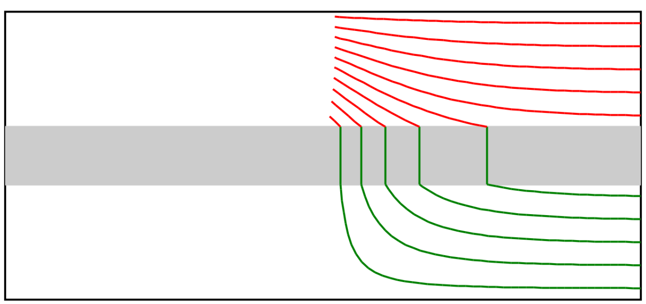
Example: $$\vec{h} = \frac{Q}{2\pi T_{tot}} \ln(r) \vec{e} + \sum_{n=1}^{N-1} \frac{A_n}{2\pi} \mathrm{K}_0(r\sqrt{w_n}) \vec{u}_n$$

(implemented formulation is in discharge potentials)

A Line-sink in top aquifer

Drawdown in top aquifer

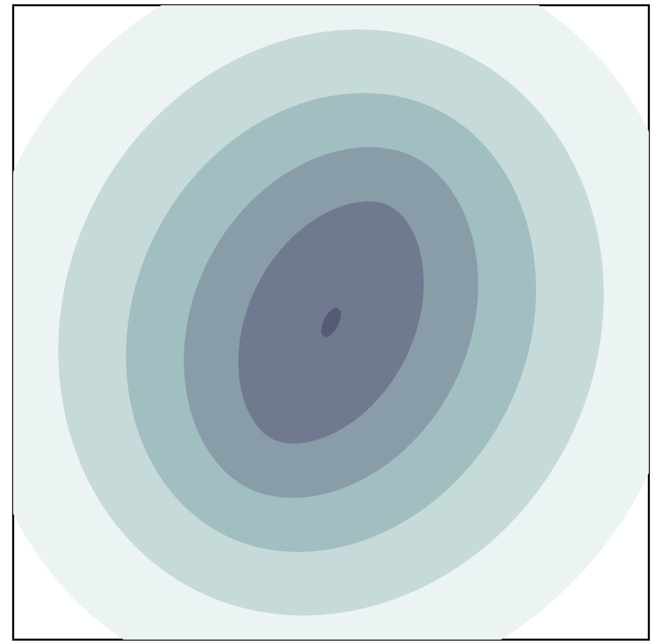




Path lines started in top aquifer

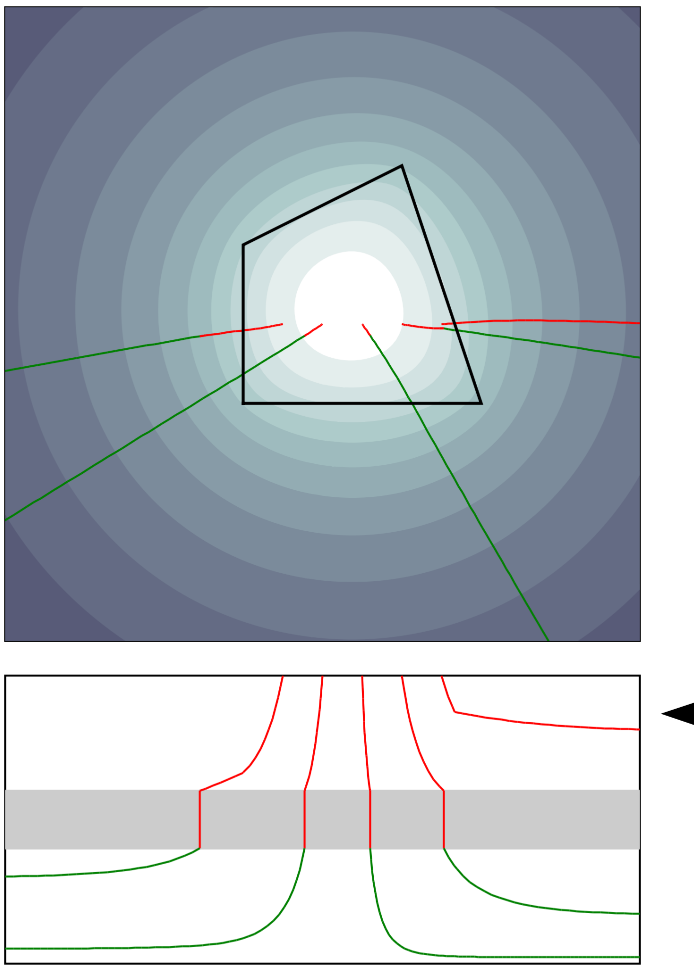
Path lines started in bottom aquifer

Drawdown bottom aquifer



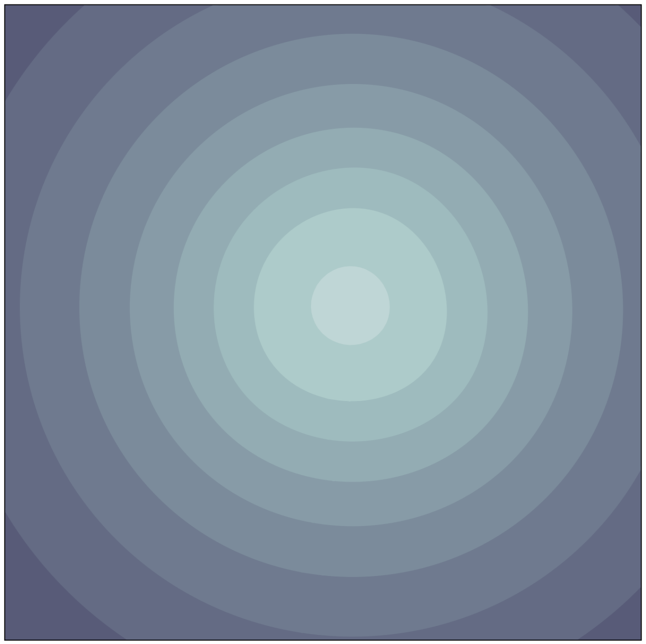
An area-sink on top of the aquifer sytem, here used for recharge

Recharge inside polygon

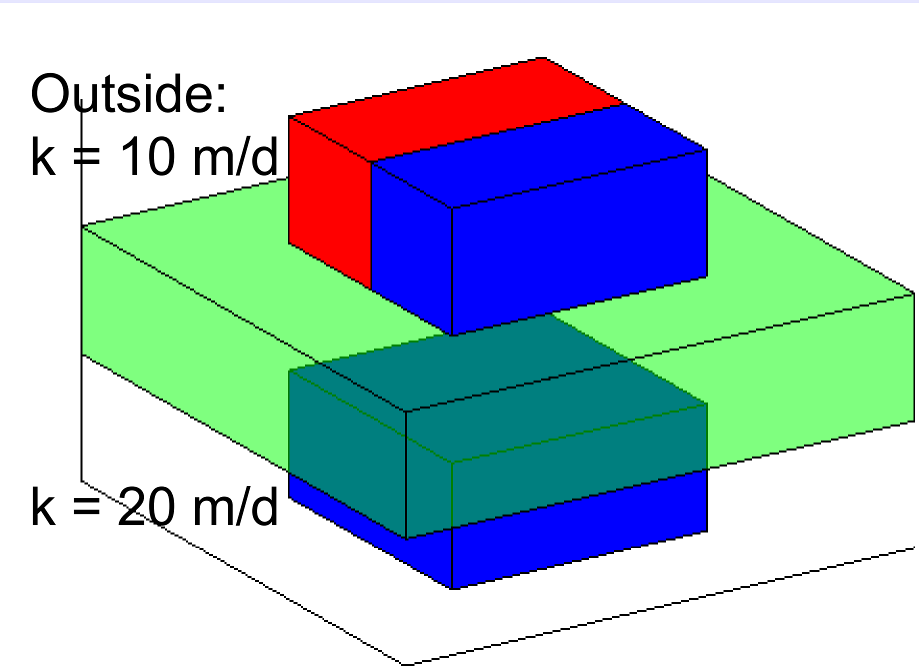


Path lines started at top of aquifer

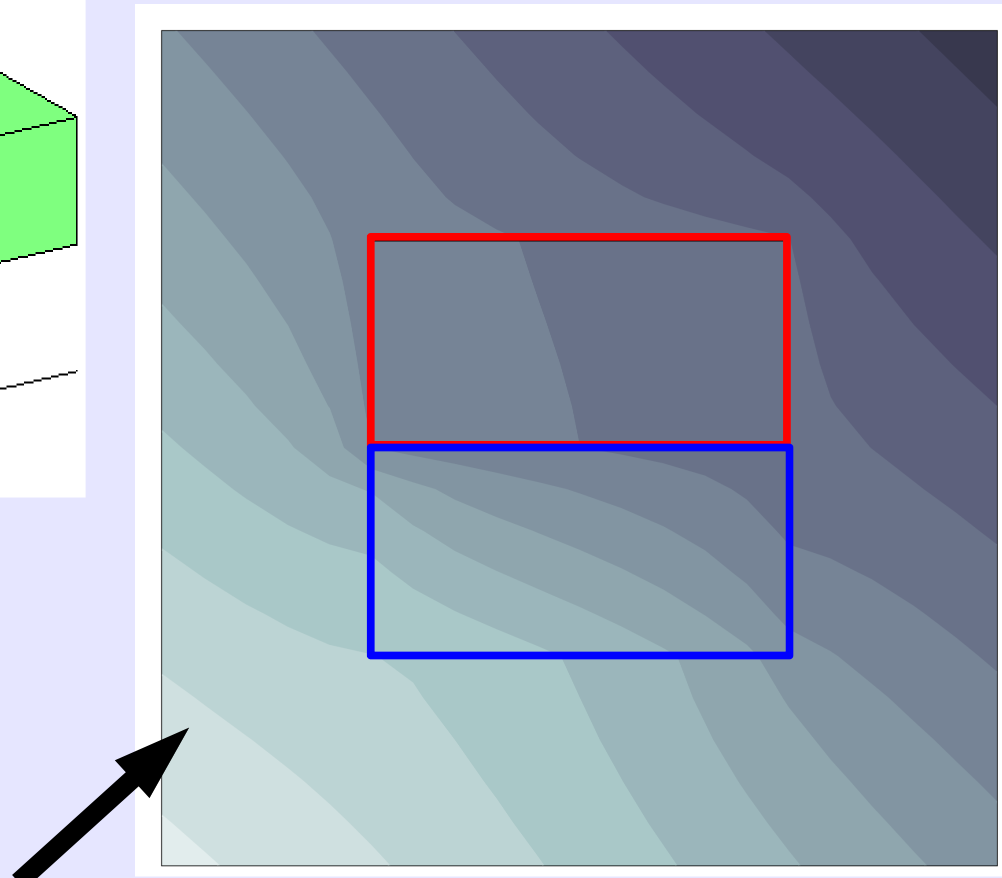
Water mount in bottom aquifer



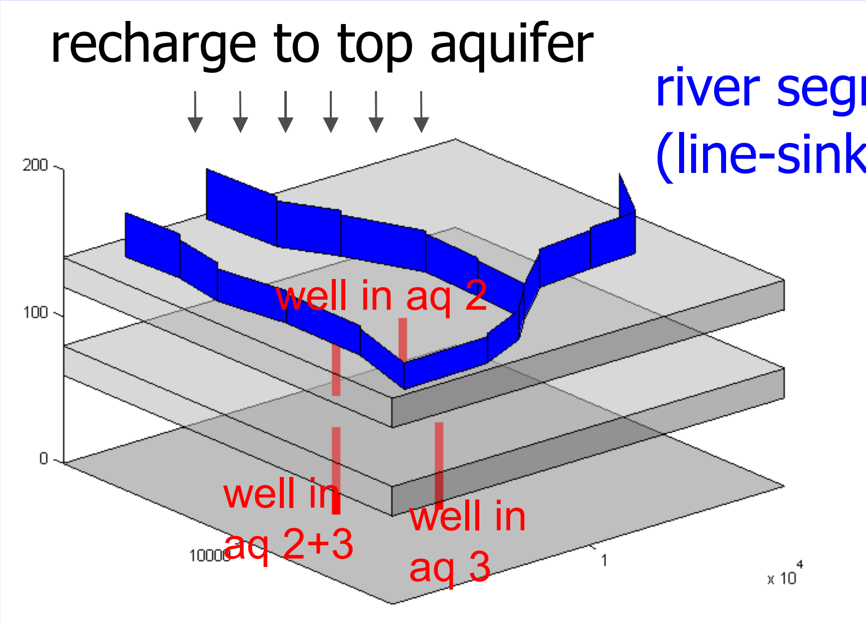
Multi-aquifer inhomogeneity in uniform flow



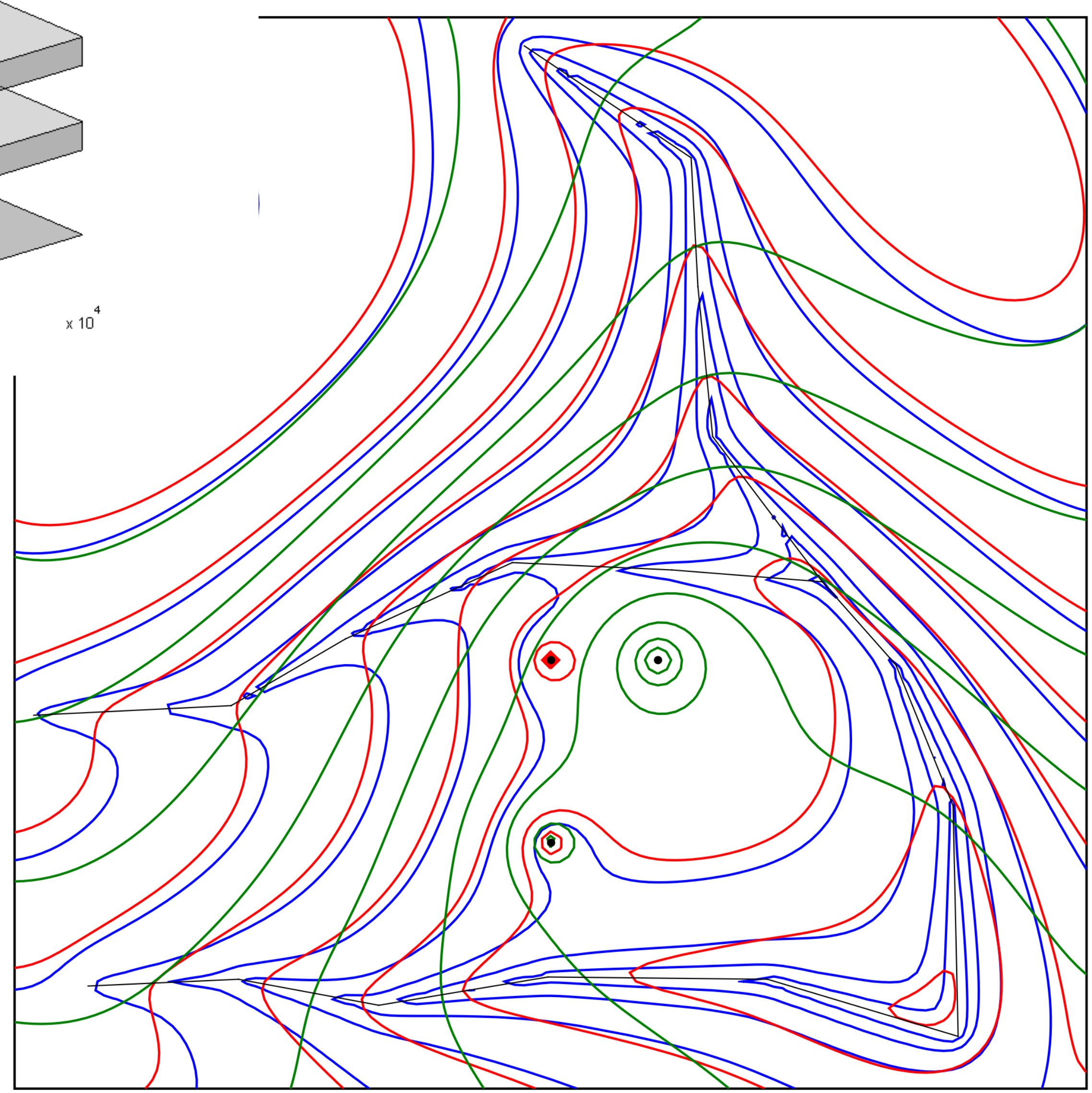
Heads in top aquifer



Red: k = 100 m/d
Blue: k = 2 m/d
Green: leaky layer

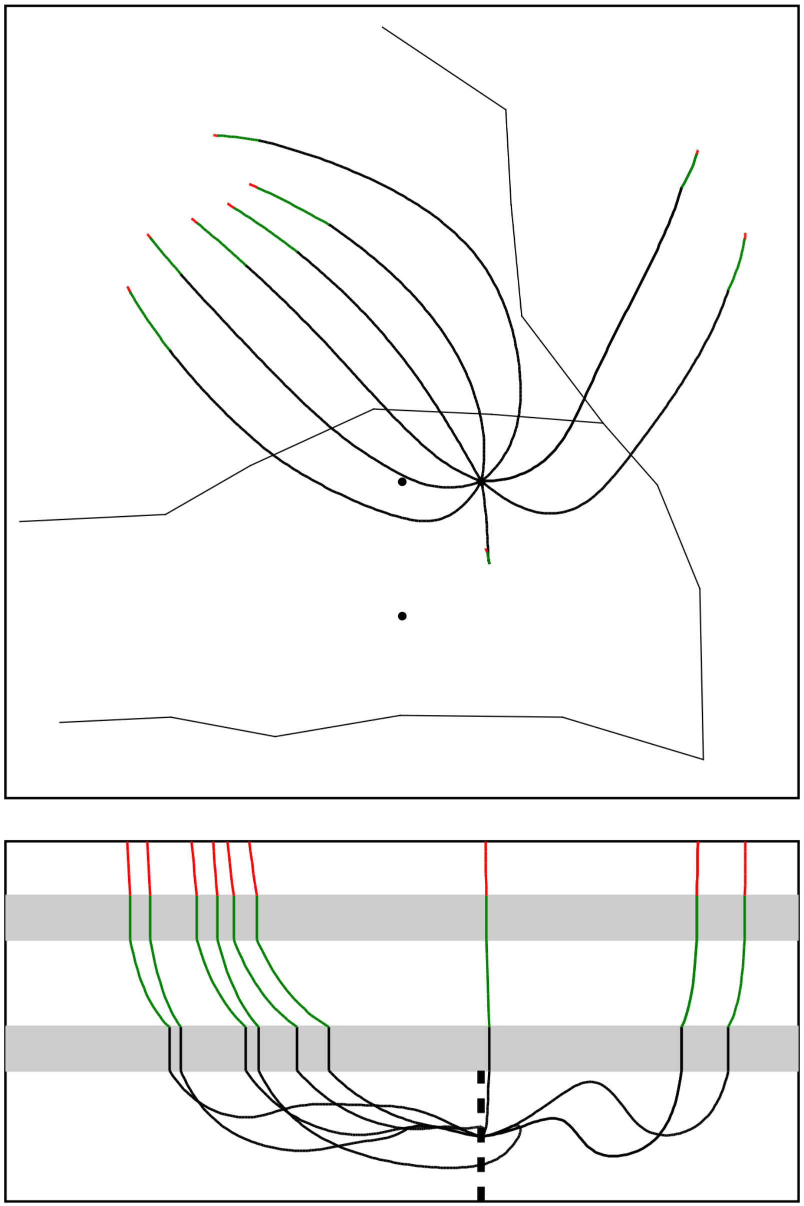


river segments (line-sinks)



Blue: Aquifer 1
Red: Aquifer 2
Green: Aquifer 3

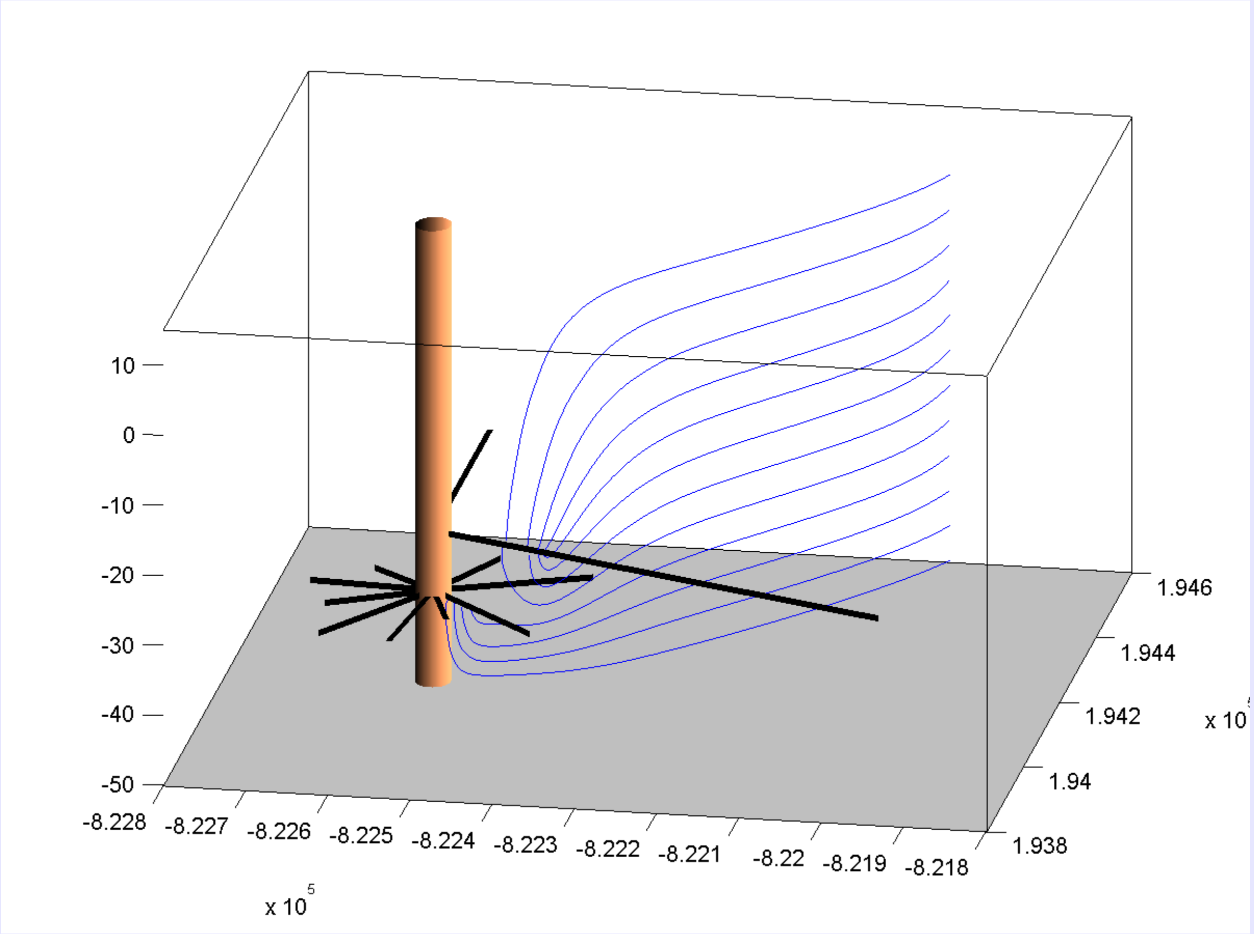
Determine source of well in aquifer 3



Path lines traced back from the well

Example application: modeling radial collector wells with a multi-layer approach

Collector well in Sonoma County, CA; model with 12 layers



WHPA, Bloomington, IN

Heterogeneity in the vadose zone causes advective spreading

Effective longitudinal spreading in saturated heterogeneous media quantified with analytic element solutions (Dagan, Fiori, Jankovic)

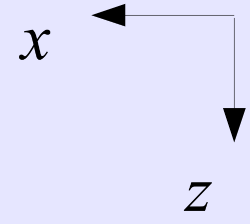
Analytic element solutions for unsaturated flow through heterogeneous vadose zones were developed recently (Bakker & Nieber 2004, VZJ, WRR)

(an open door)

Collaborator:
John Nieber, Univ. of Minnesota

Mathematical formulation

Darcy's law for specific discharge:



$$q_x = -k\frac{\partial \psi}{\partial x} \qquad q_z = -k\frac{\partial \psi}{\partial z} + k$$

ψ : pressure head (negative for unsaturated flow)

$k(\psi)$: hydraulic conductivity function

Continuity of steady flow:

$$\frac{\partial}{\partial x}\left(k\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \psi}{\partial z}\right) - \frac{\partial k}{\partial z} = 0$$

Highly non-linear deq

Kirchhoff potential and Gardner model

Gustave Kirchhoff

Kirchhoff potential: $H(\psi)=\int_{-\infty}^{\psi} k(s)\,ds$

Hydraulic conductivity:
(Gardner model) $k(\psi)=k_s \exp[\alpha(\psi-\psi_e)]$

k_s : hydraulic conductivity at saturation

α : parameter dependent on pore size distribution

ψ_e : air entry pressure head

Physically: $H(\psi)=\dfrac{k(\psi)}{\alpha}$ substitution gives ....

Kirchhoff potential is governed by linear differential equation

$$\nabla^2 H - \alpha \frac{\partial H}{\partial z} = 0$$

Define: $\Theta = H \exp[-\alpha(z - z_c)/2]$

which gives $$\nabla^2 \Theta = \frac{\alpha^2}{4} \Theta$$

Modified Helmholtz equation

Hermann von Helmholtz

Approach:

Superimpose analytic element solutions for $H(x,z)$, by using solutions to the Mod. Helmholtz Eq.

Analytic element solutions for circular and elliptical inhomogeneities

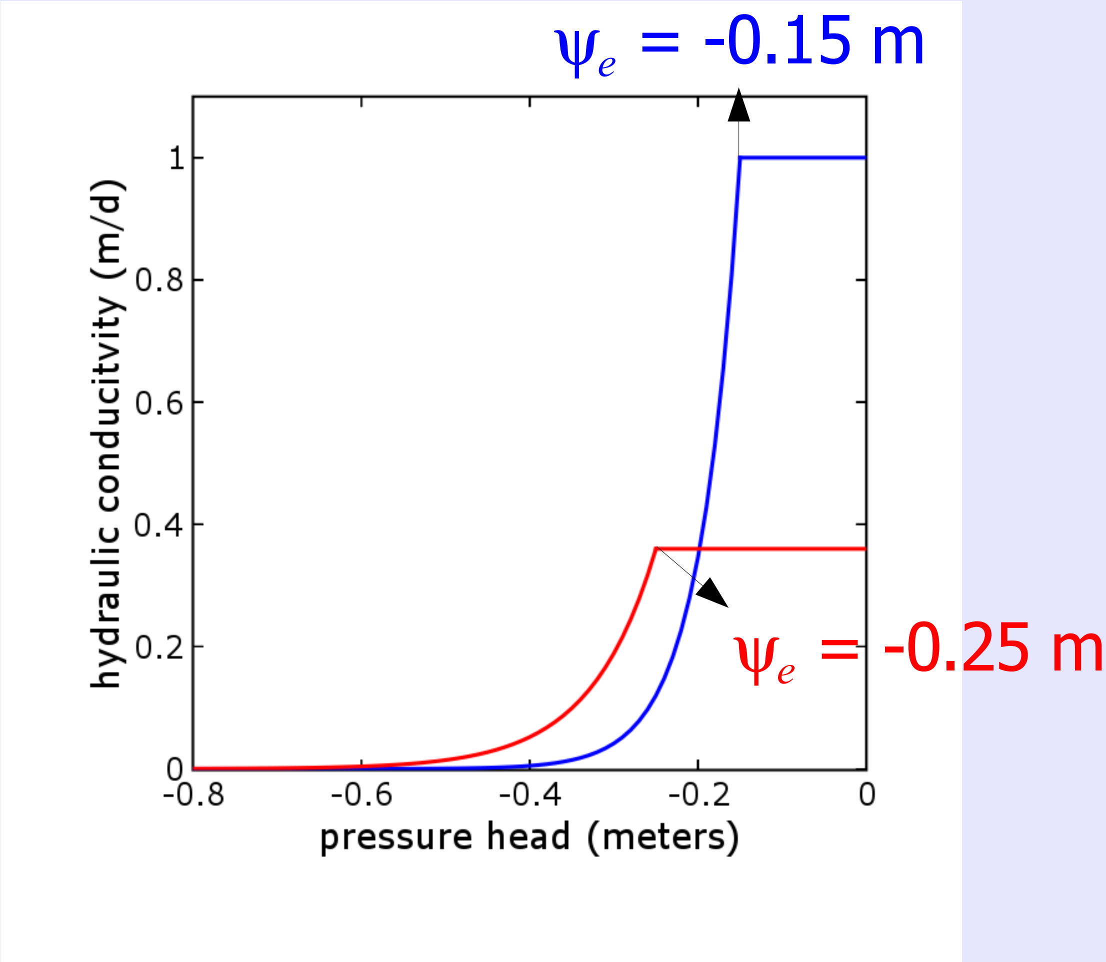
Hydraulic conductivity functions $k = k_s e^{\alpha(\psi - \psi_e)}$ are different inside and outside (k_s, α, and ψ_e may differ)

Separation of variables in radial or elliptical coordinates

Separate infinite series for inside and outside

Boundary conditions of continuity of pressure head and normal flow are met up to machine accuracy

Commonly, k of finer soil is larger than k of coarser soil under unsaturated conditions



Consider a finer-grained medium containing coarser grained inclusions

finer material

k_s = 0.36 m/d

ψ_e = -0.25 m/d

α = 12.9 m^{-1}

λ = 3.9 m^{-1}

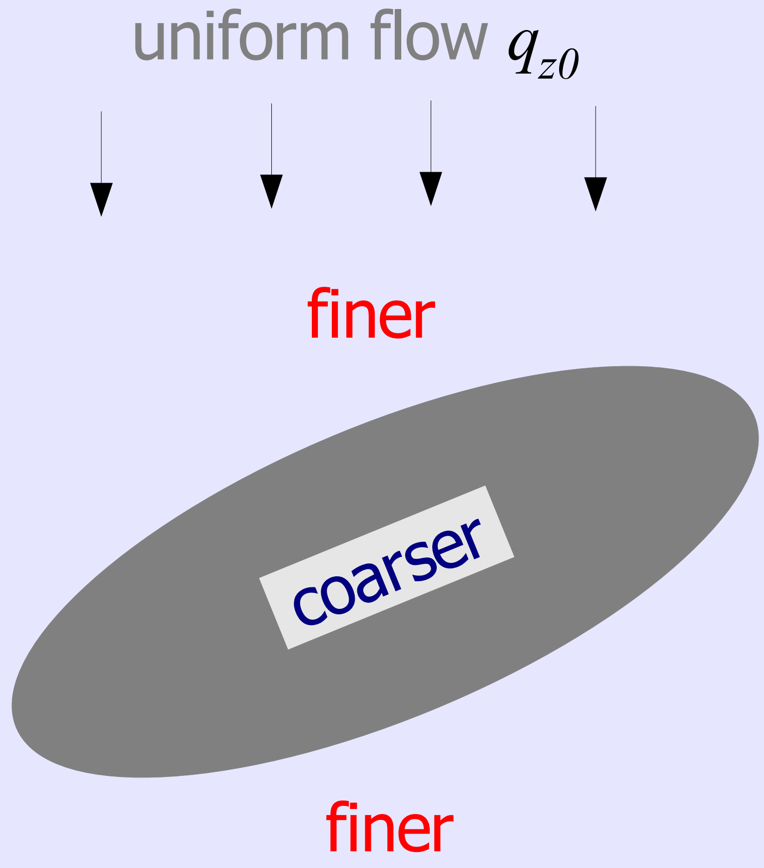
coarser material

k_s = 1 m/d

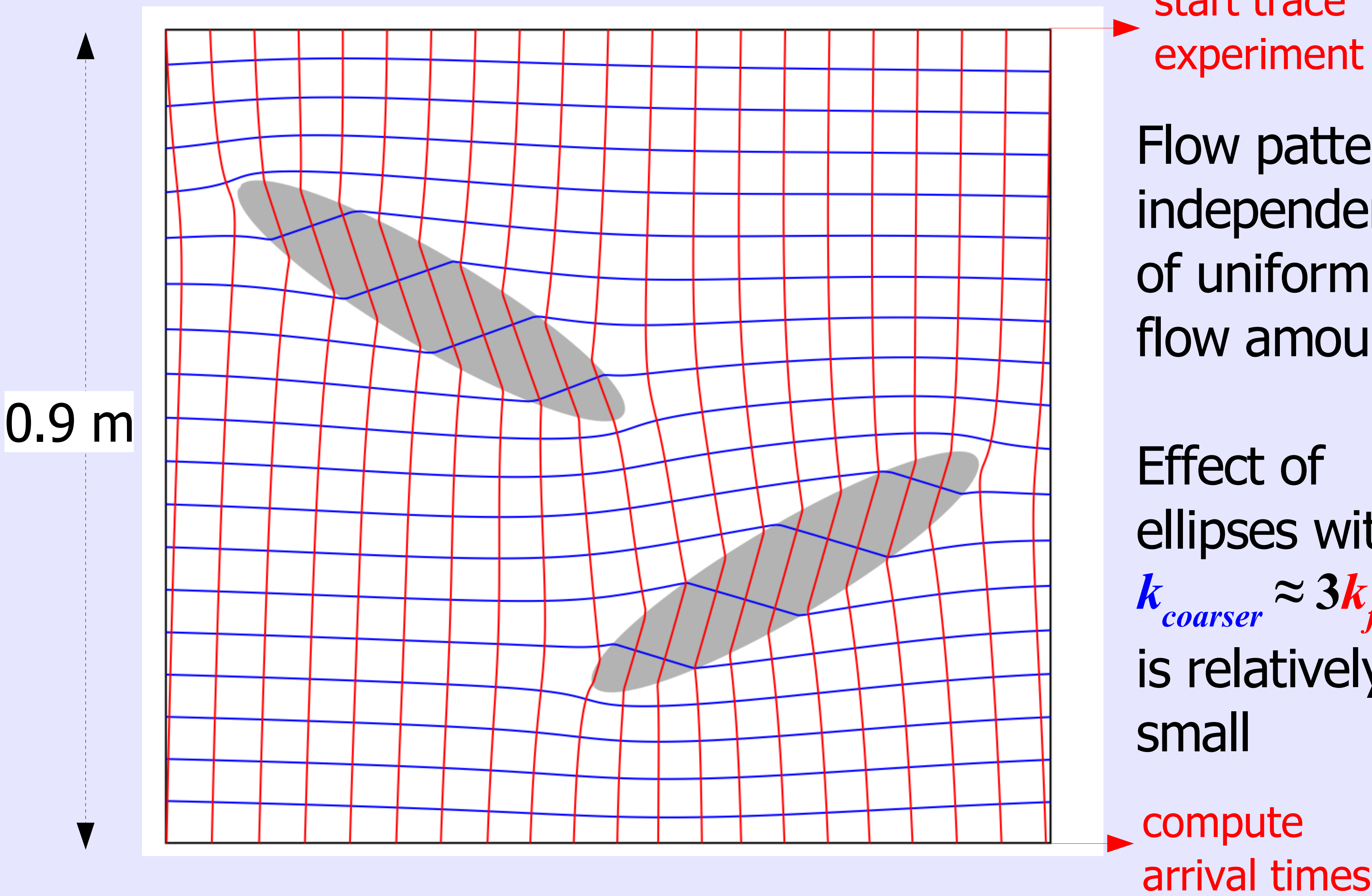
ψ_e = -0.15 m/d

α = 21.2 m^{-1}

λ = 6.2 m^{-1}



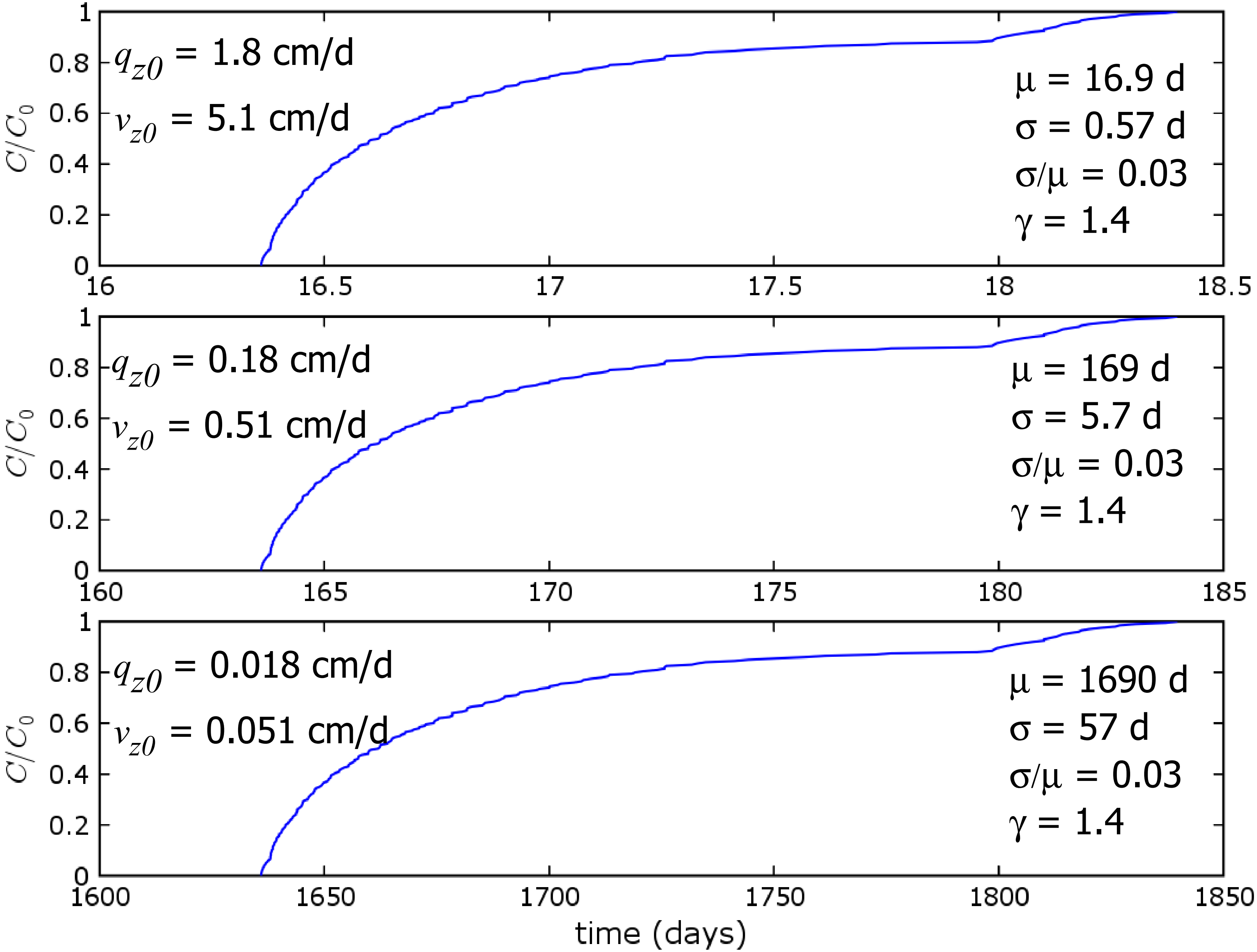
In <u>saturated</u> flow, the coarser ellipses will attract the flow



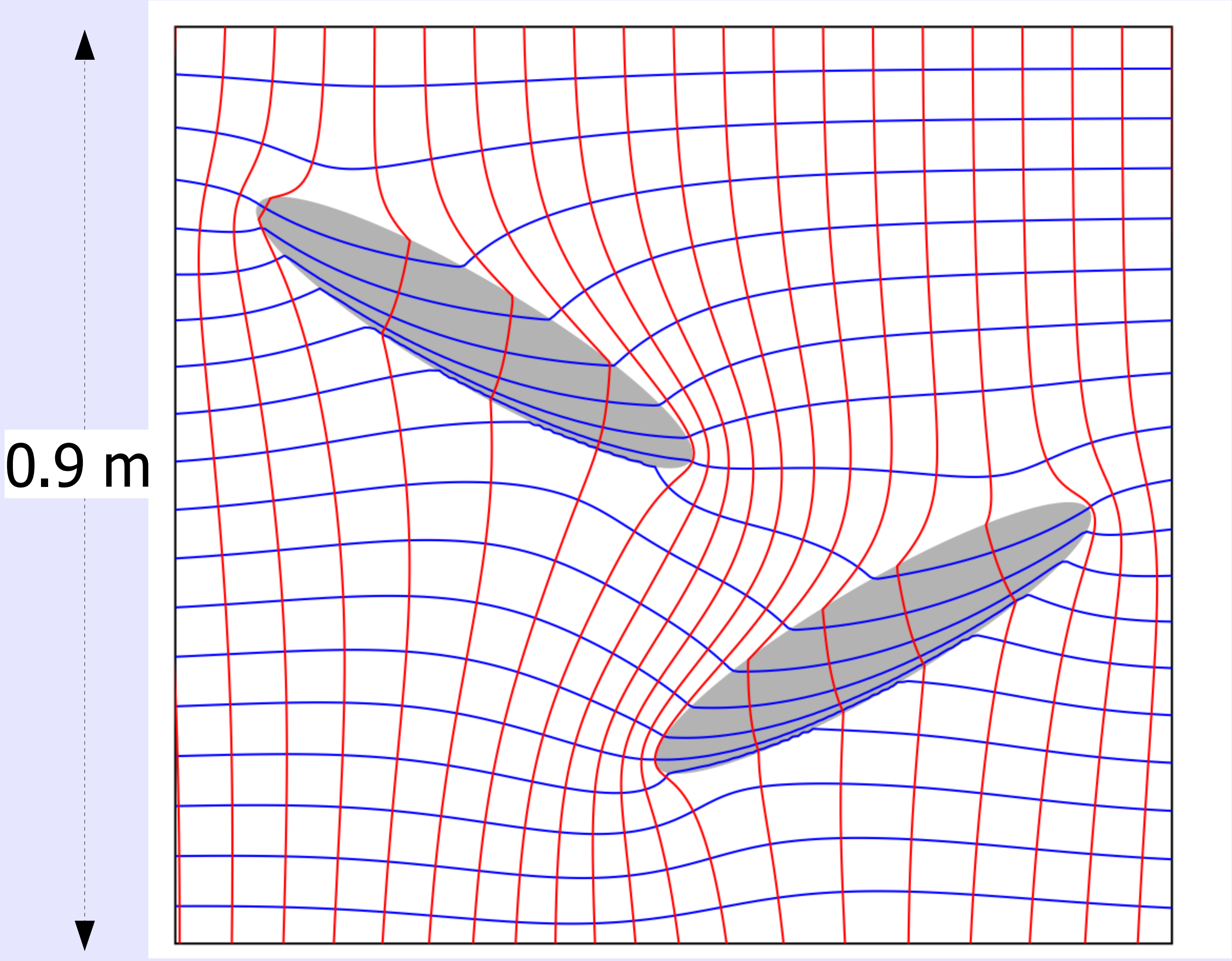
Flow pattern independent of uniform flow amount

Effect of ellipses with $k_{coarser} \approx 3k_{finer}$ is relatively small

Saturated flow: Breakthrough curve independent of uniform flow



Unsaturated flow:
coarse-grained ellipses divert flow

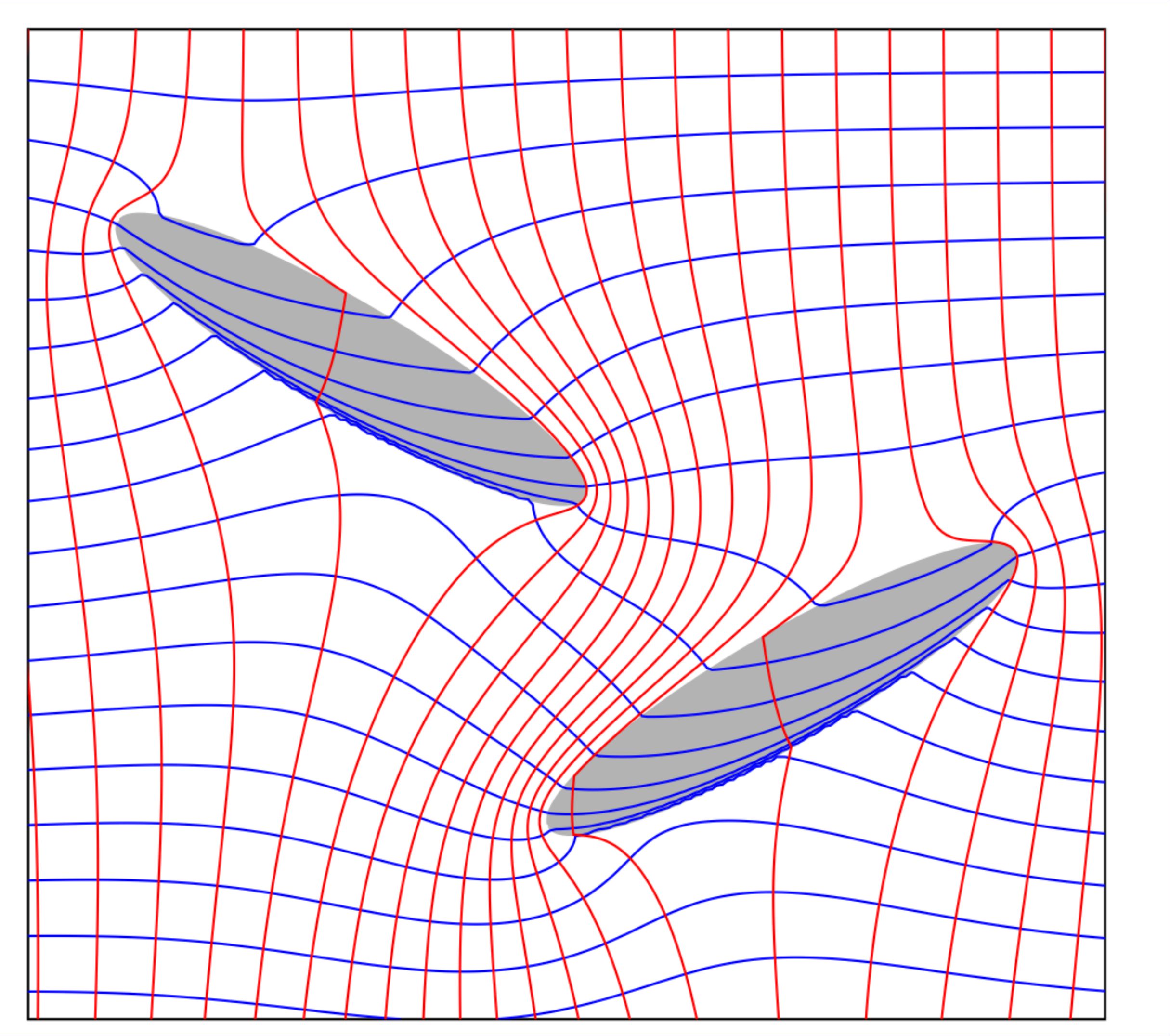


$q_{z0} = 0.05 k_{s,fine}$

$= 1.8$ cm/d

$v_{z0} = 12.8$ cm/d

A smaller uniform flow creates a greater diversion by the ellipses

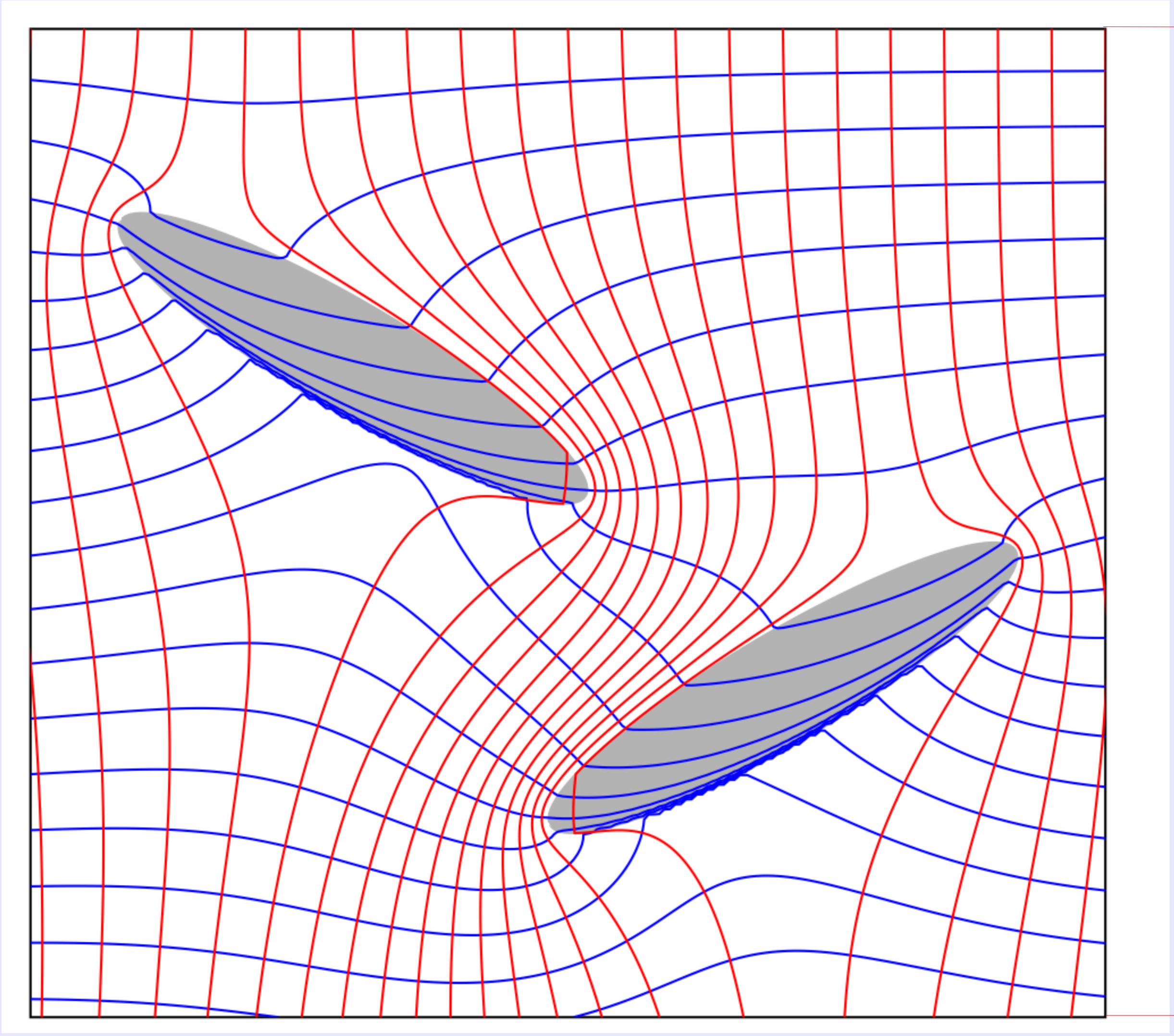


$q_{z0} = 0.005 k_{s,fine}$

$= 0.18$ cm/d

$v_{z0} = 2.56$ cm/d

Coarse-grained ellipses in very small uniform flow behave almost as impermeable ellipses



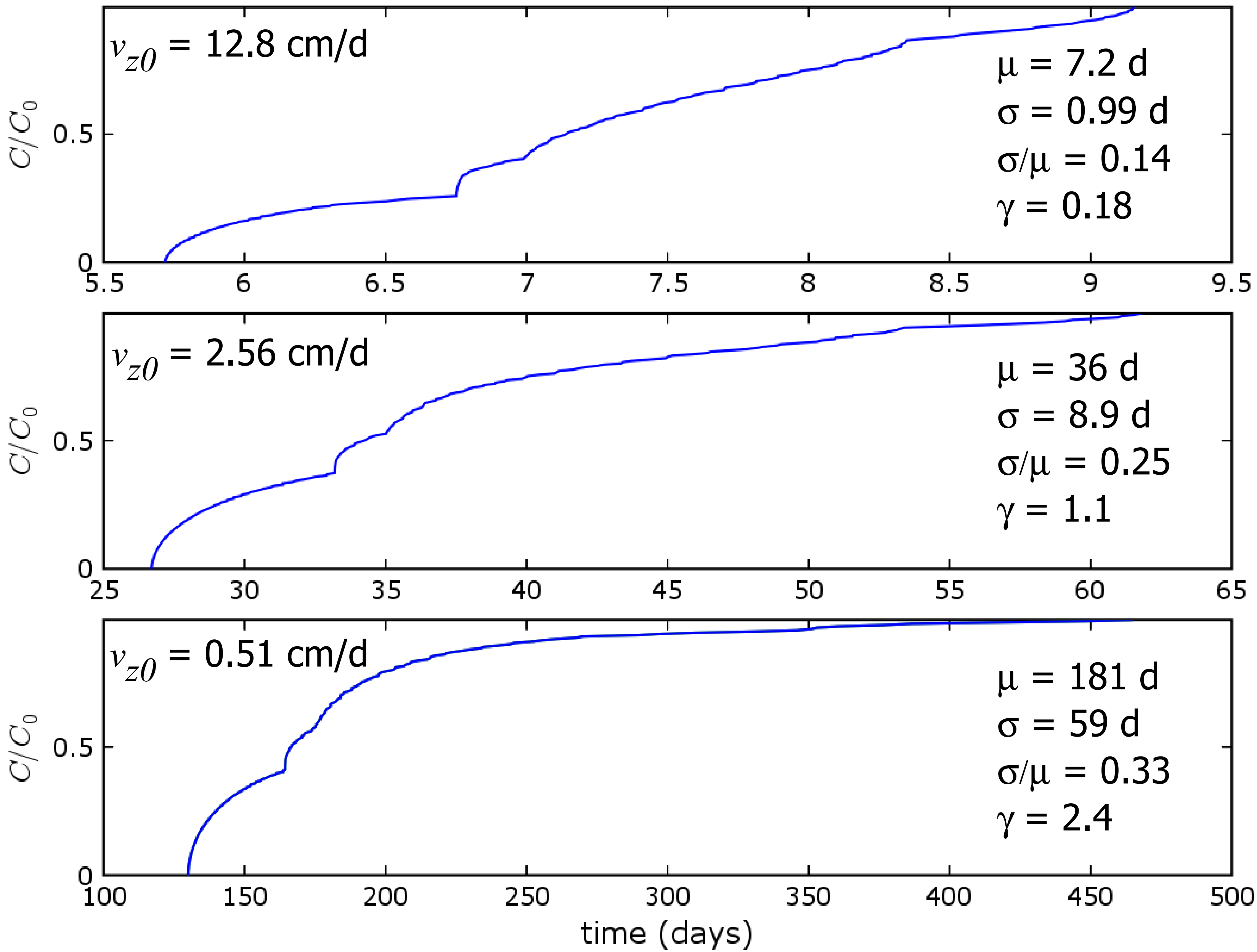
start trace experiment

$q_{z0} = 0.0005 k_{s,fine}$
$= 0.018$ cm/d

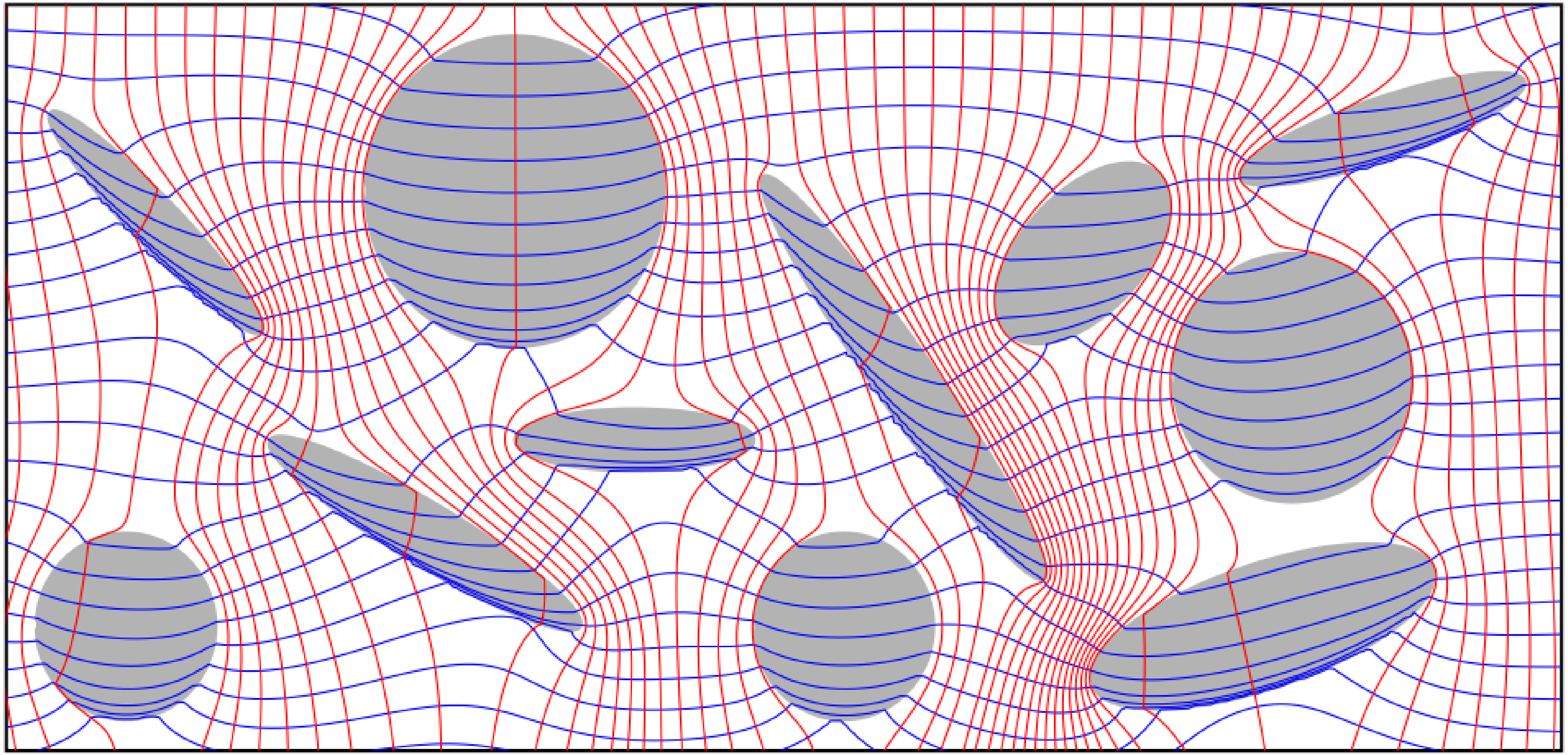
$v_{z0} = 0.51$ cm/d

compute arrival times

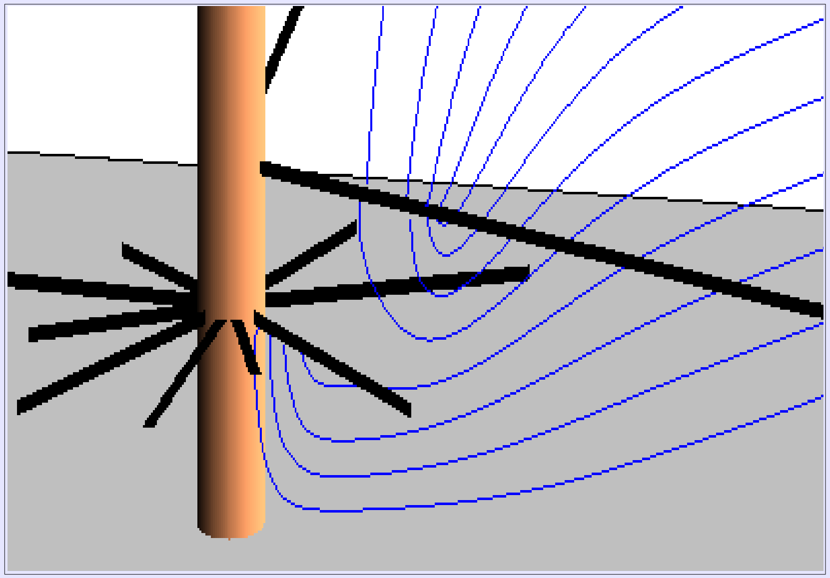
Breakthrough curves for unsaturated flow



Accurate and efficient models can be made of flow through many (thousands?) of inhomogeneities

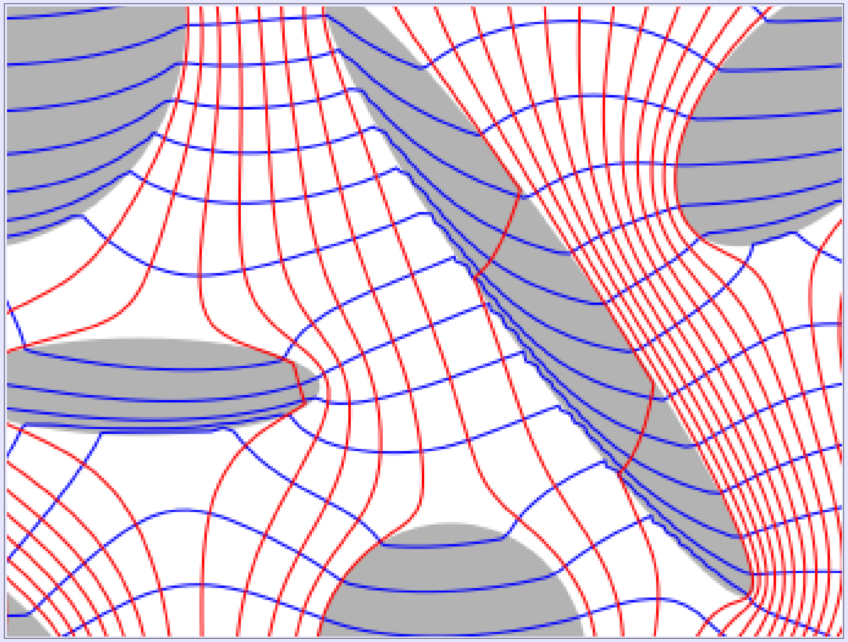


Analytic modeling of head and flow in heterogeneous multi-layer systems and vadose zones



Multi-layer solution of 3D flow to a radial collector well

Inhomogeneities in vadose zone have much greater effect than in saturated zone and benefit from analytic modeling



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